

# Subsumed homoclinic connections and infinitely many coexisting attractors in piecewise-linear maps.

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## Abstract

We establish an equivalence between infinitely many asymptotically stable periodic solutions and subsumed homoclinic connections for  $N$ -dimensional piecewise-linear continuous maps. These features arise as a codimension-three phenomenon. The periodic solutions are single-round: they each involve one excursion away from a central saddle-type periodic solution. The homoclinic connection is subsumed in the sense that one branch of the unstable manifold of the saddle solution is contained entirely within its stable manifold. The results are proved by using exact expressions for the periodic solutions and components of the stable and unstable manifolds which are available because the maps are piecewise-linear. We also describe a practical approach for finding this phenomenon in the parameter space of a map and illustrate the results with the three-dimensional border-collision normal form.

## 1 Introduction

One of the most important global features of a discrete-time dynamical system (or map) is that of a homoclinic connection. Homoclinic connections are given by intersections between the stable and unstable manifolds of an invariant set and are central to our understanding of chaos. If the intersections are transverse, then they are not broken by small perturbations to the map. For this reason, transverse homoclinic connections are robust structures. Their existence implies the presence of a Smale horseshoe, infinitely many unstable periodic orbits, and chaotic dynamics [1].

As the parameters of a smooth map are varied, transverse homoclinic connections can be created (or destroyed) in homoclinic tangencies. Here, stable and unstable manifolds intersect tangentially. A homoclinic tangency is a codimension-one phenomenon, thus to understand the nature of the dynamics associated with the tangency as the system is perturbed, it suffices to vary a single parameter.

In the simplest setting: that the invariant set is a saddle fixed point of a two-dimensional map, these dynamics were first studied rigorously in [2, 3]. The simplest nearby bifurcations are those of single-round periodic solutions (involving one excursion away from the fixed point). These bifurcations are described by a one-dimensional quadratic map through the process of renormalisation. The single-round periodic solutions can be asymptotically stable, but sufficiently close to a non-degenerate homoclinic tangency, asymptotically stable single-round periodic solutions do not coexist. Other invariant sets exist near homoclinic tangencies, such as multi-round periodic solutions, and indeed the complete bifurcation structure is fractal [4]. Attractors may coexist, and infinitely many attractors coexist on parameter sets known as Newhouse regions [5, 6].

In more than two dimensions, only the slowest stable and unstable directions about the saddle fixed point are important to the dynamics. This is because almost all forward orbits that approach the fixed point do so tangent to the slowest stable direction, and almost all backward orbits that approach the fixed point asymptotically do so tangent to the slowest unstable direction. Thus if the eigenvalues associated with these directions are real and of multiplicity one, then as in the two-dimensional case the renormalised dynamics are described by a one-dimensional quadratic map. Cases involving complex eigenvalues lead to different maps, and it has been shown that four maps suffice to describe all possible scenarios [7, 8, 9].

Various types of degenerate homoclinic tangencies have been studied. Cubic and other higher-order tangencies yield a similar basic bifurcation structure [10, 11]. A degeneracy in the global reinjection mechanism allows for the existence of infinitely many elliptic single-round periodic solutions in area-preserving maps [12, 13]. For a codimension-two scenario at which the branches of the stable and unstable manifolds that intersect are coincident, an unfolding reveals single-round periodic solutions existing between pairs of saddle-node bifurcations [14]. A codimension-three homoclinic tangency corresponding to a Shilnikov-Hopf bifurcation is unfolded in [15]. Also, for piecewise-smooth maps, homoclinic connections can be created in homoclinic corners [16].

This paper concerns piecewise-linear maps that are continuous but non-differentiable on a codimension-one manifold termed the switching manifold. The global dynamics of such maps describe the local dynamics of border-collision bifurcations [17, 18]. Such dynamics can involve multiple attractors [19, 20, 21]. This paper builds on a result of [22] for such maps in two dimensions. There it was shown that, under certain conditions, if the map has infinitely many stable single-round periodic solutions, then it must have a particular coincident homoclinic connection that is codimension-three. As the map is perturbed from this codimension-three scenario, the number of coexisting attractors scales with  $\frac{\ln(\varepsilon)}{\ln(\lambda)}$ , where  $\varepsilon$  is the size of the parameter change and  $\lambda$  is the associated stable eigenvalue [23]. Examples with the same underlying mechanism had been given earlier in an area-preserving scenario [24] and for a three-component piecewise-smooth map [25].

In this paper the result of [22] is extended to  $N$ -dimensions (Theorem 5.2). Here the unstable manifold is again one-dimensional, but the stable manifold is now  $(N-1)$ -dimensional. Branches of the stable and unstable manifolds therefore cannot be coincident. Instead we find that one branch of the unstable manifold is contained within the stable manifold — we say it is *subsumed*. We also provide a converse result (Theorem 5.1) showing that, under certain conditions, the presence of a subsumed homoclinic connection, as characterised using an orbit with two points on the switching manifold, implies the existence of infinitely many asymptotically stable single-round periodic solutions.

The remainder of this paper is organised as follows. In §2 we introduce a general piecewise-linear continuous map  $f$  and explain how orbits and periodic solutions of  $f$  are encoded symbolically. We also characterise the stability and admissibility of periodic solutions and provide a simple result regarding line segments.

In §3 we provide rationale for the assumptions placed upon the symbolic itineraries in Theorems 5.1 and 5.2. This is achieved by investigating the consequences that an orbit  $\{y_i\}$ , homoclinic to an  $\mathcal{X}$ -cycle (periodic solution with itinerary  $\mathcal{X}$ ) is of a simple type. Then in §4 we show that if the dimension of the unstable manifold of the  $\mathcal{X}$ -cycle is one and  $\{y_i\}$  has two particular points on the switching manifold of  $f$ , then, under certain conditions, the  $\mathcal{X}$ -cycle has a subsumed homoclinic connection, Proposition 4.1. In §5 we then state and discuss Theorems 5.1 and 5.2.

In §6 we explain how the codimension-three scenario can be identified numerically in a map with at least three parameters and provide three specific examples using the three-dimensional border-collision normal form. Theorems 5.1 and 5.2 are then proved in §7 using a slew of algebraic, analytical, and geometric arguments. The over-riding strategy is to work in a coordinate system centred about one point of the  $\mathcal{X}$ -cycle and with axes that at least partially align with the stable and unstable subspaces of the  $\mathcal{X}$ -cycle. Finally §8 provides a summary and outlook for future studies.

## 2 The basic properties of a general piecewise-linear map

Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  be defined by

$$f(x) = \begin{cases} A_L x + b, & e_1^\top x \leq 0, \\ A_R x + b, & e_1^\top x \geq 0, \end{cases} \quad (2.1)$$

where  $A_L$  and  $A_R$  are real-valued  $N \times N$  matrices and  $b \in \mathbb{R}^N$ . Throughout this paper,  $e_i$  denotes the  $i^{\text{th}}$  coordinate vector of  $\mathbb{R}^N$ . Thus, in particular,  $e_1^\top x$  is the first component of  $x \in \mathbb{R}^N$ .

On the hyperplane  $e_1^\top x = 0$ , which we denote by  $\Sigma$ , the function  $f$  is continuous but not differentiable (unless  $A_L = A_R$ ). The assumption of continuity of  $f$  on  $\Sigma$  implies that  $A_L$  and  $A_R$  differ in only their first columns. That is, there exists  $\xi \in \mathbb{R}^N$  such that

$$A_R = A_L + \xi e_1^\top. \quad (2.2)$$

In the context of dynamical systems,  $f$  is a piecewise-linear continuous map with switching manifold  $\Sigma$ . It is general in the sense that any piecewise-linear continuous function with two components can be put in the form (2.1) by choosing coordinates so that the boundary between the components is  $\Sigma$ . Given any  $x_0 \in \mathbb{R}^N$ , we let  $\{x_i\}$  denote the forward orbit of  $x_0$  under  $f$ , as defined by  $x_{i+1} = f(x_i)$ , for all  $i \geq 0$ .

### 2.1 Using words to encode finite parts of orbits

It is helpful to encode the itinerary of orbits of  $f$ , relative to  $\Sigma$ , using the alphabet  $\{L, R\}$ . Throughout this paper we write a word  $\mathcal{X}$  of length  $n$  as

$$\mathcal{X} = \mathcal{X}_0 \cdots \mathcal{X}_{n-1},$$

where  $\mathcal{X}_i \in \{L, R\}$  for each  $i = 0, \dots, n-1$ .

**Definition 2.1.** Given  $x_0 \in \mathbb{R}^N$  and a word  $\mathcal{X}$  of length  $n$ , we say that  $x_0$  *follows*  $\mathcal{X}$  under  $f$  if

$$\begin{aligned}\mathcal{X}_i &= L \text{ whenever } e_1^\top x_i < 0, \\ \mathcal{X}_i &= R \text{ whenever } e_1^\top x_i > 0,\end{aligned}\tag{2.3}$$

over all  $i = 0, \dots, n-1$ .

If  $x_i \in \Sigma$ , for some  $i = 0, \dots, n-1$ , then there is no restriction on  $\mathcal{X}_i$  in Definition 2.1. In this case  $x_0$  follows more than one word of length  $n$ . For example, the orbit shown in Fig. 1 has  $x_0, x_3 \in \Sigma$  and so  $x_0$  follows four words of length five under  $f$ :

$$LLLLR, \quad LLLRR, \quad RLLLL, \quad RLLRR.\tag{2.4}$$

Here we formalise this observation. Throughout this paper  $\mathcal{X}^{\bar{i}}$  denotes the word formed from  $\mathcal{X}$  by flipping the symbol  $\mathcal{X}_i$  (from  $L$  to  $R$ , or vice-versa).

**Lemma 2.1.** *Suppose  $x_0$  follows  $\mathcal{X}$  under  $f$  and  $x_i \in \Sigma$  for some  $i = 0, \dots, n-1$ . Then  $x_0$  also follows  $\mathcal{X}^{\bar{i}}$  under  $f$ .*

## 2.2 Line segments

We use  $\mathcal{L}$  to denote line segments in  $\mathbb{R}^N$ , with square brackets when endpoints are included in the segment and round brackets otherwise. For instance, given  $y, z \in \mathbb{R}^N$ ,

$$\mathcal{L}[y, z] = \{(1-s)y + sz \mid 0 \leq s \leq 1\}.$$

If  $y$  and  $z$  lie on the same side of  $\Sigma$ , then each  $x \in \mathcal{L}[y, z]$  also lies on this side of  $\Sigma$ . In this case, the image of  $\mathcal{L}[y, z]$  under  $f$  is given using only one component of  $f$ . Both components of  $f$  are affine, hence this image is another line segment. By repeating this observation and extending it to allow points on  $\Sigma$  we obtain the following result.

**Lemma 2.2.** *Suppose  $y_0$  and  $z_0$  follow  $\mathcal{X}$  under  $f$ . Then every  $x_0 \in \mathcal{L}[y_0, z_0]$  also follows  $\mathcal{X}$  under  $f$ .*

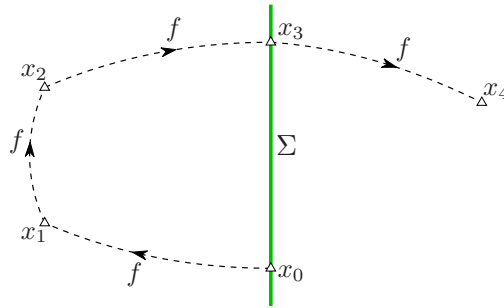


Figure 1: A sketch of the initial part of the forward orbit of a point  $x_0 \in \mathbb{R}^N$  that follows each of the words in (2.4) under the map  $f$ .

## 2.3 Periodic solutions

Let

$$f_L(x) = A_L x + b, \quad f_R(x) = A_R x + b,$$

denote the two components of  $f$ .

**Definition 2.2.** An  $\mathcal{X}$ -cycle is an  $n$ -tuple,  $\{x_i^\mathcal{X}\}$ , for which

$$f_{\mathcal{X}_0}(x_0^\mathcal{X}) = x_1^\mathcal{X}, \quad f_{\mathcal{X}_1}(x_1^\mathcal{X}) = x_2^\mathcal{X}, \quad \dots, \quad f_{\mathcal{X}_{n-1}}(x_{n-1}^\mathcal{X}) = x_0^\mathcal{X}. \quad (2.5)$$

If  $x_0^\mathcal{X}$  follows  $\mathcal{X}$  under  $f$ , then the  $\mathcal{X}$ -cycle is in fact an orbit of  $f$  and said to be *admissible*. The next result is an immediate consequence of Lemma 2.1.

**Lemma 2.3.** Suppose  $\{x_i^\mathcal{X}\}$  is an  $\mathcal{X}$ -cycle and  $x_j^\mathcal{X} \in \Sigma$  for some  $j = 0, \dots, n-1$ . Then  $\{x_i^\mathcal{X}\}$  is also an  $\mathcal{X}^{\bar{j}}$ -cycle.

For example, the periodic solution shown in Fig. 2 has two points on  $\Sigma$  and is an  $\mathcal{X}$ -cycle where  $\mathcal{X}$  is any of the four words (2.4).

## 2.4 Existence, uniqueness, and stability of periodic solutions

Let

$$f_\mathcal{X} = f_{\mathcal{X}_{n-1}} \circ \dots \circ f_{\mathcal{X}_0}$$

represent the composition of  $f_L$  and  $f_R$  in the order specified by  $\mathcal{X}$ . The point  $x_0^\mathcal{X}$  of an  $\mathcal{X}$ -cycle is a fixed point of  $f_\mathcal{X}$ . Furthermore, each  $x_i^\mathcal{X}$  is a fixed point of  $f_{\mathcal{X}^{(i)}}$ , where throughout this paper we use  $\mathcal{X}^{(i)}$  to denote the  $i^{\text{th}}$  left cyclic permutation of  $\mathcal{X}$ , that is

$$\mathcal{X}^{(i)} = \mathcal{X}_i \dots \mathcal{X}_{n-1} \mathcal{X}_0 \dots \mathcal{X}_{i-1}.$$

The function  $f_\mathcal{X}$  is affine, and

$$f_\mathcal{X}(x) = M_\mathcal{X} x + P_\mathcal{X} b,$$

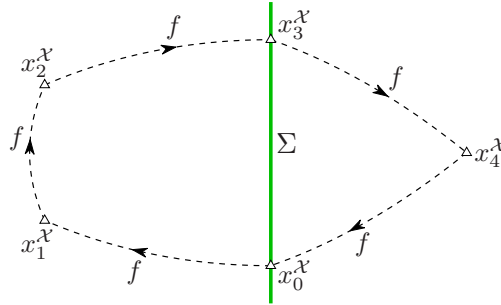


Figure 2: A periodic solution of  $f$  that is an  $\mathcal{X}$ -cycle for any of the words in (2.4).

where

$$M_{\mathcal{X}} = A_{\mathcal{X}_{n-1}} \cdots A_{\mathcal{X}_0} , \quad (2.6)$$

$$P_{\mathcal{X}} = I + A_{\mathcal{X}_{n-1}} + A_{\mathcal{X}_{n-1}}A_{\mathcal{X}_{n-2}} + \cdots + A_{\mathcal{X}_{n-1}} \cdots A_{\mathcal{X}_1} . \quad (2.7)$$

If 1 is not an eigenvalue of  $M_{\mathcal{X}}$ , then the  $\mathcal{X}$ -cycle is unique and

$$x_i^{\mathcal{X}} = (I - M_{\mathcal{X}^{(i)}})^{-1} P_{\mathcal{X}^{(i)}} b , \quad \text{for all } i = 0, \dots, n-1 .$$

If the  $\mathcal{X}$ -cycle is admissible with no points on  $\Sigma$ , then every point in some open region containing  $x_0^{\mathcal{X}}$  follows  $\mathcal{X}$  under  $f$ . The  $\mathcal{X}$ -cycle is therefore a stable periodic solution of  $f$  if and only if all eigenvalues of  $M_{\mathcal{X}}$  have modulus less than or equal to 1. The  $\mathcal{X}$ -cycle is furthermore an asymptotically stable periodic solution of  $f$  if and only if all eigenvalues of  $M_{\mathcal{X}}$  have modulus strictly less than 1. If instead the  $\mathcal{X}$ -cycle involves points on  $\Sigma$ , then its stability is much more difficult to characterise [26].

## 2.5 Using symbol sequences to encode orbits

We encode orbits of  $f$  with bi-infinite sequences

$$\mathcal{S} = \cdots \mathcal{S}_{-1} \mathcal{S}_0 \mathcal{S}_1 \cdots ,$$

where  $\mathcal{S}_i \in \{L, R\}$  for all  $i \in \mathbb{Z}$ .

**Definition 2.3.** An  $\mathcal{S}$ -orbit is a sequence  $\{x_i^{\mathcal{S}}\}$  for which

$$x_{i+1}^{\mathcal{S}} = f_{\mathcal{S}_i}(x_i^{\mathcal{S}}) ,$$

for all  $i \in \mathbb{Z}$ .

Admissibility of  $\mathcal{S}$ -orbits parallels that of  $\mathcal{X}$ -cycles. Specifically, if an  $\mathcal{S}$ -orbit satisfies (2.3) for all  $i \in \mathbb{Z}$  (using  $\mathcal{S}$  in place of  $\mathcal{X}$ ), then it is an orbit of  $f$  and we say it is admissible.

## 3 Symbolic representations for simple homoclinic connections

In this section we convert elementary observations regarding homoclinic connections into formal results stated in terms of the words and sequences introduced in §2. These results motivate the assumptions contained in Theorems 5.1 and 5.2 and establish some restrictions on the words that can be used in these theorems.

Let  $\mathcal{X}$  be a word of length  $n$ , let  $\{x_i^{\mathcal{X}}\}$  be an admissible  $\mathcal{X}$ -cycle, and let  $\{y_i\}$  be an orbit homoclinic to  $\{x_i^{\mathcal{X}}\}$ . That is, the sequence  $\{y_i\}$  converges to the  $\mathcal{X}$ -cycle as  $i \rightarrow \pm\infty$ . Fig. 3 shows an example.

The orbit  $\{y_i\}$  is an  $\mathcal{S}$ -orbit, where

$$\mathcal{S} = \mathcal{X}^{\infty} \mathcal{Y} \mathcal{X}^{\infty} , \quad (3.1)$$

for some word  $\mathcal{Y}$  of length  $p$ . For simplicity we assume that  $y_0$  corresponds to the symbol  $\mathcal{Y}_0$ . Equation (3.1) is short-hand for the precise description

$$\mathcal{S}_i = \begin{cases} \mathcal{X}_{i \bmod n} , & i < 0 , \\ \mathcal{Y}_i , & i = 0, \dots, p-1 , \\ \mathcal{X}_{(i-p) \bmod n} , & i \geq p , \end{cases} \quad (3.2)$$

where, for any  $i \in \mathbb{Z}$ , we write  $i \bmod n$  to denote the integer in  $\{0, \dots, n-1\}$  that differs from  $i$  by an integer multiple of  $n$ . We also assume  $\mathcal{X}_0 \neq \mathcal{Y}_0$  so that as we follow the points of the  $\mathcal{S}$ -orbit in order, the word  $\mathcal{Y}$  starts at the first point at which periodicity following  $\mathcal{X}$  ceases.

We now look at the behaviour of the  $\mathcal{S}$ -orbit more closely. For each  $i = 0, \dots, n-1$ , let

$$\gamma_i = \bigcup_{j \in \mathbb{Z}} \mathcal{L}[y_{(j-1)n+i}, y_{jn+i}] , \quad (3.3)$$

as shown in Fig. 3. Let us first discuss  $\gamma_0$ . By (3.2), as  $j \rightarrow -\infty$  the sequence  $\{y_{jn}\}$  converges to  $x_0^{\mathcal{X}}$ . Similarly, as  $j \rightarrow \infty$ , the sequence  $\{y_{jn}\}$  converges to  $x_d^{\mathcal{X}}$ , where

$$d = -p \bmod n . \quad (3.4)$$

Therefore  $\gamma_0$  is a path from  $x_0^{\mathcal{X}}$  to  $x_d^{\mathcal{X}}$ . Similarly each  $\gamma_i$  is a path from  $x_i^{\mathcal{X}}$  to  $x_{(d+i) \bmod n}^{\mathcal{X}}$ .

Let

$$\mathcal{P} = \bigcup_{i=0}^{n-1} \gamma_i \cup x_i^{\mathcal{X}}$$

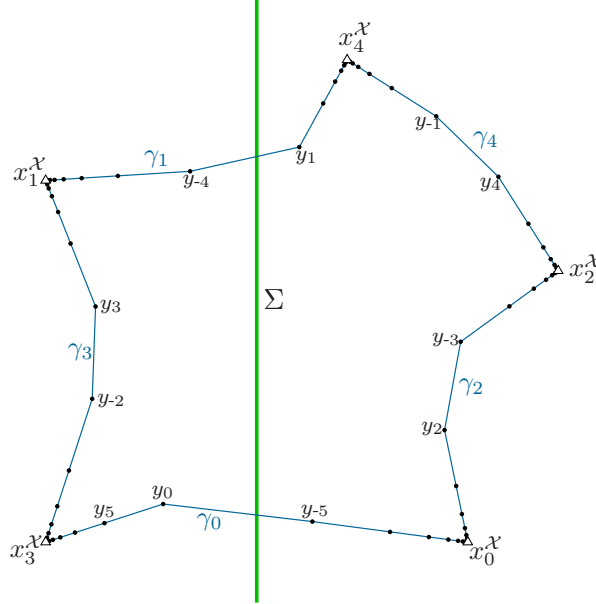


Figure 3: A sketch of the phase space of  $f$  showing an orbit  $\{y_i\}$  homoclinic to an  $\mathcal{X}$ -cycle, where  $\mathcal{X} = RLRLR$ . The orbit is an  $\mathcal{S}$ -orbit, where  $\mathcal{S}$  is given by (3.1) and  $\mathcal{Y} = LR$ . The piecewise-linear paths  $\gamma_0, \dots, \gamma_4$ , (3.3), are also shown. Notice,  $\gamma_0$  is a path from  $x_0^{\mathcal{X}}$  to  $x_3^{\mathcal{X}}$  because  $d = 3$ , as given by (3.4) with  $n = 5$  and  $p = 2$  (the lengths of  $\mathcal{X}$  and  $\mathcal{Y}$ ).

denote the union of the paths and the  $\mathcal{X}$ -cycle. In general,  $\mathcal{P}$  is a collection of loops involving the points of the  $\mathcal{X}$ -cycle. If

$$d \neq 0, \quad \gcd(d, n) = 1, \quad (3.5)$$

where  $\gcd$  abbreviates greatest common divisor, then  $\mathcal{P}$  consists of a single loop, as in Fig. 3.

Next we look for where  $\mathcal{P}$  intersects  $\Sigma$ . Since  $\mathcal{X}_0 \neq \mathcal{Y}_0$ , we have  $\mathcal{S}_{-n} \neq \mathcal{S}_0$ . Assuming for simplicity that  $y_{-n}$  and  $y_0$  do not lie on  $\Sigma$ , they therefore lie on different sides of  $\Sigma$ . Thus the line segment  $\mathcal{L}[y_{-n}, y_0)$  of  $\mathcal{P}$  intersects  $\Sigma$ . Since  $\mathcal{P}$  is a collection of loops, it must intersect  $\Sigma$  elsewhere. We now consider the simplest scenario: that  $\mathcal{P}$  intersects  $\Sigma$  at only one other point.

More precisely, we suppose that the symbols  $\mathcal{S}_{(j-1)n+i}$  and  $\mathcal{S}_{jn+i}$ , that correspond to the endpoints of  $[y_{(j-1)n+i}, y_{jn+i}]$ , differ only for  $i = j = 0$  and one other pair of values, call them  $\hat{i}$  and  $\hat{j}$ . That is,

$$\mathcal{S}_{(j-1)n+i} \neq \mathcal{S}_{jn+i} \text{ if and only if } i = j = 0 \text{ or } i = \hat{i}, j = \hat{j}. \quad (3.6)$$

In view of (3.2), we must have  $\hat{j} \geq 0$ . Assuming  $\hat{i} \neq 0$  (which is the case whenever  $\mathcal{X}$  contains both  $L$ 's and  $R$ 's) the elements of  $\mathcal{S}$  are given in terms of the elements of  $\mathcal{X}$  by

$$\begin{aligned} \mathcal{S}_{jn} &= \mathcal{X}_0 \text{ if and only if } j < 0, \\ \mathcal{S}_{jn+\hat{i}} &= \mathcal{X}_{\hat{i}} \text{ if and only if } j < \hat{j}, \\ \mathcal{S}_{jn+i} &= \mathcal{X}_i \text{ otherwise.} \end{aligned} \quad (3.7)$$

**Lemma 3.1.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be words of length  $n$  and  $p$  with  $\mathcal{X}_0 \neq \mathcal{Y}_0$ . Suppose  $\mathcal{S}$ , as given by (3.1), satisfies (3.6). Then*

$$\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\bar{0}\bar{\alpha}}, \quad (3.8)$$

where  $\alpha = \hat{j}n + \hat{i}$ .

The identity (3.8) says that if we concatenate  $\mathcal{Y}$  and  $\mathcal{X}$  to form the word  $\mathcal{Y}\mathcal{X}$  (of length  $n+p$ ), then flip the symbols for indices 0 and  $\alpha$ , we obtain the concatenation of  $\mathcal{Y}$  and  $\mathcal{X}$  in the reverse order. Note that we always have  $\alpha < n+p$  in view of (3.2).

*Proof.* Choose any  $k = 0, \dots, n+p-1$ . Then  $k = jn+i$  for unique values  $j \in \mathbb{Z}$  and  $i \in \{0, \dots, n-1\}$ . Since  $\mathcal{S}_0$  corresponds to  $\mathcal{Y}_0$ , we have  $(\mathcal{Y}\mathcal{X})_k = \mathcal{S}_k = \mathcal{S}_{jn+i}$ , and  $(\mathcal{X}\mathcal{Y})_k = \mathcal{S}_{k-n} = \mathcal{S}_{(j-1)n+i}$ . By (3.6),  $(\mathcal{X}\mathcal{Y})_k = (\mathcal{Y}\mathcal{X})_k$  if and only if  $k \neq 0, \alpha$ . Hence  $\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\bar{0}\bar{\alpha}}$  as required.  $\square$

The identity  $\mathcal{X}^{(d)} = \mathcal{X}^{\bar{0}\bar{\hat{i}}}$  (where  $\mathcal{X}^{(d)}$  is the  $d^{\text{th}}$  left cyclic permutation of  $\mathcal{X}$ ) follows immediately from (3.6). Here we show that this identity is more generally a consequence of (3.8).

**Lemma 3.2.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be words of length  $n$  and  $p$  that satisfy (3.8) for some  $\alpha \in \{1, \dots, n+p-1\}$ , and let  $d$  be given by (3.4). Then*

$$\mathcal{X}^{(d)} = \mathcal{X}^{\bar{0}\bar{\hat{i}}}, \quad (3.9)$$

where  $\hat{i} = \alpha \bmod n$ .

*Proof.* Choose any  $i \in \{0, \dots, n-1\}$ . Then  $\mathcal{X}_i = (\mathcal{X}\mathcal{Y})_i = (\mathcal{Y}\mathcal{X})_i$  unless  $i = 0$  in which case  $\mathcal{X}_i = (\mathcal{X}\mathcal{Y})_i \neq (\mathcal{Y}\mathcal{X})_i$ . Let  $k$  be the unique integer for which  $p \leq kn+i < p+n$ . From  $(\mathcal{Y}\mathcal{X})_i$  we now repeatedly step  $n$  places along  $\mathcal{Y}\mathcal{X}$  until we reach  $(\mathcal{Y}\mathcal{X})_{kn+i}$  as follows. For each



$j = 1, \dots, k$ , we have  $(\mathcal{Y}\mathcal{X})_{(j-1)n+i} = \mathcal{Y}_{(j-1)n+i} = (\mathcal{X}\mathcal{Y})_{jn+i} = (\mathcal{Y}\mathcal{X})_{jn+i}$  unless  $i = \hat{i}$  and  $j = \hat{j}$  (which arises because  $\hat{j}n + \hat{i} < p + n$ ) in which case  $(\mathcal{X}\mathcal{Y})_{jn+i} \neq (\mathcal{Y}\mathcal{X})_{jn+i}$ . Since  $kn + i \geq p$ , we have  $(\mathcal{Y}\mathcal{X})_{kn+i} = \mathcal{X}_{kn+i-p} = \mathcal{X}_{(i+d) \bmod n} = \mathcal{X}_i^{(d)}$ . We have thus shown that  $\mathcal{X}_i = \mathcal{X}_i^{(d)}$  if and only if  $i \notin \{0, \hat{i}\}$ , except in the special case  $\hat{i} = 0$  for which we have shown that  $\mathcal{X}_i = \mathcal{X}_i^{(d)}$  for all  $i$ . This verifies (3.9).  $\square$

Lastly we note that words satisfying (3.9) are “rotational”, as defined below, if  $d$  satisfies (3.5). Such words are well-studied in the context of piecewise-linear maps because they relate to rigid rotation on a circle [27, 18].

**Definition 3.1.** Let  $\ell$ ,  $m$  and  $n$  be positive integers with

$$\ell < n, \quad m < n, \quad \gcd(m, n) = 1.$$

Define a word  $\mathcal{F}[\ell, m, n]$  of length  $n$  by

$$\mathcal{F}[\ell, m, n] = \begin{cases} L, & im \bmod n < \ell, \\ R, & im \bmod n \geq \ell, \end{cases}$$

for  $i = 0, \dots, n-1$ . Words  $\mathcal{F}[\ell, m, n]$  and their cyclic permutations are called *rotational*.

**Lemma 3.3.** Let  $\mathcal{X}$  be a word of length  $n$  with  $\mathcal{X}_0 = R$ . Suppose (3.9) is satisfied for some  $\hat{i} = 1, \dots, n-1$  and  $d$  satisfying (3.5). Let  $m$  denote the multiplicative inverse of  $d$  modulo  $n$  (i.e.  $md \bmod n = 1$ ) and let  $\ell = \hat{i}m \bmod n$ . Then  $\mathcal{X}^{(d)} = \mathcal{F}[\ell, m, n]$ .

For a proof of Lemma 3.3 see [28].

## 4 Subsumed homoclinic connections

Let  $\mathcal{X}$  be a word of length  $n$ . Suppose that  $\lambda_1$  and  $\lambda_2$  are eigenvalues of  $M_{\mathcal{X}}$  with multiplicity one. Let  $\beta$  denote the maximum modulus of the remaining eigenvalues of  $M_{\mathcal{X}}$  and suppose

$$0 \leq \beta < \lambda_2 < 1 < \lambda_1. \quad (4.1)$$

Since 1 is not an eigenvalue of  $M_{\mathcal{X}}$ , the  $\mathcal{X}$ -cycle is unique. If it is admissible with no points on  $\Sigma$ , then as a periodic solution of  $f$  it has a one-dimensional unstable manifold  $W^u(\{x_i^{\mathcal{X}}\})$  and an  $(N-1)$ -dimensional stable manifold  $W^s(\{x_i^{\mathcal{X}}\})$ .

We now consider these manifolds as they emanate from  $x_0^{\mathcal{X}}$ . For the affine map  $f_{\mathcal{X}}$ , let  $E^u(x_0^{\mathcal{X}})$  and  $E^s(x_0^{\mathcal{X}})$  denote the unstable and stable manifolds of its fixed point  $x_0^{\mathcal{X}}$ . Then for the piecewise-linear map  $f$ , the part of  $W^u(\{x_i^{\mathcal{X}}\})$  emanating from  $x_0^{\mathcal{X}}$  coincides with  $E^u(x_0^{\mathcal{X}})$ , and the part of  $W^s(\{x_i^{\mathcal{X}}\})$  emanating from  $x_0^{\mathcal{X}}$  coincides with  $E^s(x_0^{\mathcal{X}})$ . Note that  $W^u(\{x_i^{\mathcal{X}}\})$  has two branches; these emanate from the  $\mathcal{X}$ -cycle in opposite directions.

For  $i = 1, 2$ , let  $\omega_i^{\top}$  and  $\zeta_i$  be left and right eigenvectors of  $M_{\mathcal{X}}$  corresponding to  $\lambda_i$ , and with

$$\omega_i^{\top} \zeta_i = 1, \quad (4.2)$$

which can always be achieved because  $\lambda_1$  and  $\lambda_2$  have multiplicity one. Then  $E^u(x_0^{\mathcal{X}})$  has direction  $\zeta_1$  and  $E^s(x_0^{\mathcal{X}})$  includes direction  $\zeta_2$ . Moreover,  $\omega_1^{\top} \zeta_2 = 0$  and  $\omega_2^{\top} \zeta_1 = 0$  (because  $\lambda_1 \neq \lambda_2$  and

with  $i, j \in \{1, 2\}$  we have  $(\lambda_i \omega_i^\top) \zeta_j = \omega_i^\top M_{\mathcal{X}} \zeta_j = \omega_i^\top (\lambda_j \zeta_j)$  and so  $(\lambda_i - \lambda_j) \omega_i^\top \zeta_j = 0$ . Thus  $E^u(x_0^\mathcal{X})$  is orthogonal to  $\omega_2$  and  $E^s(x_0^\mathcal{X})$  is orthogonal to  $\omega_1$ . Since  $\beta < \lambda_2$ , direction  $\zeta_2$  represents the slowest direction for the dynamics within  $E^s(x_0^\mathcal{X})$ .

If  $e_1^\top \zeta_1 \neq 0$ , then  $E^u(x_0^\mathcal{X})$  intersects  $\Sigma$  at a unique point (denoted  $y_0$  below), see Fig. 4.

**Proposition 4.1.** *Suppose*

- i)  $\mathcal{X}$  and  $\mathcal{Y}$  satisfy  $\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\bar{0}\bar{\alpha}}$  for some  $\alpha = 1, \dots, n + p - 1$ ;
- ii) the eigenvalues of  $M_{\mathcal{X}}$  satisfy (4.1), and  $e_1^\top \zeta_1 \neq 0$ ;
- iii) the  $\mathcal{X}$ -cycle is admissible with  $x_0^\mathcal{X} \notin \Sigma$ ;
- iv) there exists an admissible  $\mathcal{S}$ -orbit, denoted  $\{y_i\}$  and where  $\mathcal{S}$  is given by (3.1), that is homoclinic to the  $\mathcal{X}$ -cycle and with  $y_0 = E^u(x_0^\mathcal{X}) \cap \Sigma$  and  $y_\alpha \in \Sigma$ .

Then

$$W_{\text{br}}^u(\{x_i^\mathcal{X}\}) = \bigcup_{i \in \mathbb{Z}} \mathcal{L}[y_i, y_{i+n}) \subset W^s(\{x_i^\mathcal{X}\}),$$

where  $W_{\text{br}}^u(\{x_i^\mathcal{X}\})$  denotes the branch of  $W^u(\{x_i^\mathcal{X}\})$  that contains the  $\mathcal{S}$ -orbit.

Proposition 4.1 tells us that if  $f$  has a homoclinic  $\mathcal{S}$ -orbit  $\{y_i\}$ , with  $y_0, y_\alpha \in \Sigma$ , then, under certain conditions, the branch of  $W^u(\{x_i^\mathcal{X}\})$  containing  $\{y_i\}$  is given simply by connecting each  $y_i$  to  $y_{i+n}$  by a line segment, and that this branch is contained within  $W^s(\{x_i^\mathcal{X}\})$  and so the homoclinic connection is subsumed.

Proposition 4.1 is proved below by first using Lemma 2.1 to show that since  $y_0, y_\alpha \in \Sigma$ , both  $y_0$  and  $y_n$  follow  $\mathcal{Y}\mathcal{X}^\infty$  under  $f$  and follow  $\mathcal{X}^\infty$  under backwards iteration of  $f$ . It follows that the endpoints of each interval  $[y_i, y_{i+n}]$  cannot lie on different sides of  $\Sigma$ , and the result is then a consequence of Lemma 2.2.

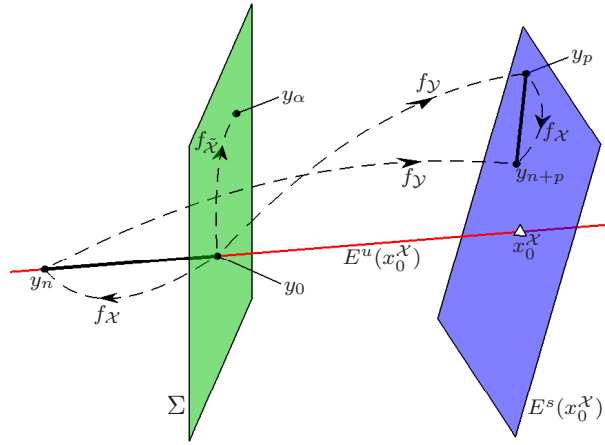


Figure 4: A sketch of the phase space of  $f$  relating to Proposition 4.1. The manifolds  $E^s(x_0^\mathcal{X})$  and  $E^u(x_0^\mathcal{X})$  are the stable and unstable manifolds of  $x_0^\mathcal{X}$  for  $f_{\mathcal{X}}$ . We show several important points of the homoclinic  $\mathcal{S}$ -orbit  $\{y_i\}$  and how they are arrived at from  $y_0$  under compositions of the components of  $f$ , where  $\mathcal{X}$  denotes the first  $\alpha$  symbols of  $\mathcal{X}\mathcal{Y}$ .

*Proof of Proposition 4.1.* By assumption,  $y_0$  follows  $\mathcal{Y}\mathcal{X}^\infty$  under  $f$  (meaning  $y_0$  follows  $\mathcal{Y}\mathcal{X}^k$  under  $f$  for all positive  $k$ ). Since  $y_0 \in \Sigma$  and  $y_\alpha \in \Sigma$ , by Lemma 2.1  $y_0$  also follows  $(\mathcal{Y}\mathcal{X})^{\bar{0}\bar{\alpha}}\mathcal{X}^\infty$ . Thus by (3.8),  $y_0$  follows  $\mathcal{X}\mathcal{Y}\mathcal{X}^\infty$ . Hence  $y_n$  follows  $\mathcal{Y}\mathcal{X}^\infty$ .

In particular, we have shown that  $y_0$  follows  $\mathcal{X}$ . Since the  $\mathcal{X}$ -cycle is admissible,  $x_0^\mathcal{X}$  also follows  $\mathcal{X}$ . Thus by Lemma 2.2, every  $x \in \mathcal{L}[x_0^\mathcal{X}, y_0]$  follows  $\mathcal{X}$ . Note that  $\mathcal{L}[x_0^\mathcal{X}, y_0]$  does not consist of a single point because  $y_0 \in \Sigma$  and  $x_0^\mathcal{X} \notin \Sigma$  and so  $y_0 \neq x_0^\mathcal{X}$ . We also have  $y_{-n}, y_{-2n}, \dots \in \mathcal{L}[x_0^\mathcal{X}, y_0]$  with  $y_{jn} \rightarrow x_0^\mathcal{X}$  as  $j \rightarrow -\infty$ .

Since  $f_\mathcal{X}$  is affine,  $W_{\text{br}}^u(\{x_i^\mathcal{X}\})$  is linear in a neighbourhood of the  $\mathcal{X}$ -cycle. Since  $y_{jn} \rightarrow x_0^\mathcal{X}$  as  $j \rightarrow -\infty$ , there exists a constant  $C \in \mathbb{Z}$  such that

$$W_{\text{br}}^u(\{x_i^\mathcal{X}\}) = \bigcup_{i=0,1,\dots} f^i(\mathcal{L}(x_0^\mathcal{X}, y_{jn})), \quad (4.3)$$

for all  $j \leq C$ . Since every  $x \in \mathcal{L}[x_0^\mathcal{X}, y_0]$  follows  $\mathcal{X}$ , we can use  $j = 1$  in (4.3). Thus, in particular,  $\mathcal{L}[y_0, y_n] \subset W_{\text{br}}^u(\{x_i^\mathcal{X}\})$ , see Fig. 4. To then describe  $W_{\text{br}}^u(\{x_i^\mathcal{X}\})$  more succinctly, we observe that  $\mathcal{L}[y_0, y_n]$  is a *fundamental domain* in that every orbit of  $f$  in  $W_{\text{br}}^u(\{x_i^\mathcal{X}\})$  involves exactly one point in  $\mathcal{L}[y_0, y_n]$ . For all  $i < 0$ , we have

$$f(\mathcal{L}[y_{i-1}, y_{i-1+n}]) = \mathcal{L}[y_i, y_{i+n}], \quad (4.4)$$

because the components of  $f$  are affine and each  $y_0, y_{-n}, \dots$  follows  $\mathcal{X}$ . Equation (4.4) also holds for all  $i \geq 0$  because  $y_0$  and  $y_n$  both follow  $\mathcal{Y}\mathcal{X}^\infty$ . Therefore  $W_{\text{br}}^u(\{x_i^\mathcal{X}\}) = \bigcup_{i \in \mathbb{Z}} \mathcal{L}[y_i, y_{i+n}]$ . The line segments  $\mathcal{L}[y_i, y_{i+n}]$  converge to the  $\mathcal{X}$ -cycle as  $i \rightarrow \infty$ , thus the forward orbit of every point in  $W_{\text{br}}^u(\{x_i^\mathcal{X}\})$  converges to the  $\mathcal{X}$ -cycle, and therefore  $W_{\text{br}}^u(\{x_i^\mathcal{X}\}) \subset W^s(\{x_i^\mathcal{X}\})$ .  $\square$

## 5 Main results

Here we state the main results and then discuss the assumptions contained within them. The results concern  $\mathcal{X}^k\mathcal{Y}$ -cycles where  $k \geq 0$ . These are periodic solutions that follow  $\mathcal{X}$  a total of  $k$  times, then follow  $\mathcal{Y}$ , and then repeat. Theorem 5.1 provides sufficient conditions for the existence of admissible, asymptotically stable  $\mathcal{X}^k\mathcal{Y}$ -cycles for infinitely many values of  $k \geq 0$ . Theorem 5.2 indicates that many of the conditions of Theorem 5.1 are necessary. Moreover, Proposition 4.1 tells us that the conditions of Theorem 5.1 imply the presence of a subsumed homoclinic connection. The nature of  $\mathcal{X}^k\mathcal{Y}^{\bar{0}}$ -cycles is also included in Theorem 5.2. These periodic solutions are unstable; numerical computations suggest that their stable manifolds typically form the boundaries between the basins of attraction of the  $\mathcal{X}^k\mathcal{Y}$ -cycles.

**Theorem 5.1.** *Suppose*

- i)  $\mathcal{X}$  and  $\mathcal{Y}$  satisfy  $\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\bar{0}\bar{\alpha}}$  for some  $\alpha = 1, \dots, n + p - 1$ ;
- ii) the eigenvalues of  $M_\mathcal{X}$  satisfy (4.1), and  $e_1^\top \zeta_1 \neq 0$ ;
- iii)  $\lambda_1 \lambda_2 = 1$  and  $\lambda_2 < c < 1$ , where

$$c = \det \left( \begin{bmatrix} \omega_1^\top \\ \omega_2^\top \end{bmatrix} M_\mathcal{Y} \begin{bmatrix} \zeta_1 & \zeta_2 \end{bmatrix} \right); \quad (5.1)$$

- iv) the  $\mathcal{X}$ -cycle is admissible with no points on  $\Sigma$ ;
- v) there exists an admissible  $\mathcal{S}$ -orbit, denoted  $\{y_i\}$  and where  $\mathcal{S}$  is defined by (3.1), that is homoclinic to the  $\mathcal{X}$ -cycle and with  $y_0 = E^u(x_0^\mathcal{X}) \cap \Sigma$  and  $y_\alpha \in \Sigma$ ;
- vi) there does not exist  $i \geq 0$  such that  $y_i \in \Sigma$  and  $y_{i+n} \in \Sigma$ .

Then there exists  $k_{\min} \geq 0$  such that for all  $k \geq k_{\min}$  the map  $f$  has an admissible, asymptotically stable  $\mathcal{X}^k\mathcal{Y}$ -cycle with no points on  $\Sigma$ .

**Theorem 5.2.** *Suppose*

- i)  $\mathcal{X}$  and  $\mathcal{Y}$  satisfy  $\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\bar{0}\bar{\alpha}}$  for some  $\alpha = 1, \dots, n+p-1$ ;
- ii) the eigenvalues of  $M_\mathcal{X}$  satisfy (4.1), and  $e_1^\top \zeta_1 \neq 0$ ;
- iii)  $c \notin \{-1, 0\}$ , where  $c$  is given by (5.1);
- iv)  $f_{\mathcal{Y}^{\bar{0}}}(E^u(x_0^\mathcal{X})) \not\subset E^s(x_0^\mathcal{X})$ ;
- v) there exists  $k_{\min} \geq 0$  such that for all  $k \geq k_{\min}$  the map  $f$  has an admissible, stable  $\mathcal{X}^k\mathcal{Y}$ -cycle with no points on  $\Sigma$ .

Then the  $\mathcal{X}$ -cycle is admissible with  $x_0^\mathcal{X} \notin \Sigma$  and there exists an  $\mathcal{S}$ -orbit (not necessarily admissible), denoted  $\{y_i\}$  and where  $\mathcal{S}$  is defined by (3.1), that is homoclinic to the  $\mathcal{X}$ -cycle and with  $y_0 = E^u(x_0^\mathcal{X}) \cap \Sigma$  and  $y_\alpha \in \Sigma$ . Moreover,  $\lambda_1\lambda_2 = 1$  and  $\lambda_2 \leq c \leq 1$ . Also,  $\mathcal{X}^k\mathcal{Y}^{\bar{0}}$ -cycles are unique (although not necessarily admissible) for sufficiently large values of  $k$ , and  $x_{kn}^{\mathcal{X}^k\mathcal{Y}^{\bar{0}}} \rightarrow y_0$  as  $k \rightarrow \infty$ .

We conclude this section with some technical remarks about the theorem statements. Proofs of the theorems are deferred to §7.

- The scenario described in Theorem 5.1 is codimension-three because each of the conditions  $y_0 \in W^s(\{x_i^\mathcal{X}\})$ ,  $y_\alpha \in \Sigma$  and  $\lambda_1\lambda_2 = 1$  is codimension-one.
- Assumptions (4) and (6) of Theorem 5.1 are included in order to prove admissibility of the  $\mathcal{X}^k\mathcal{Y}$ -cycles. If  $x_i^\mathcal{X} \in \Sigma$ , for some  $i = 1, \dots, n$ , then it may be possible for points of an  $\mathcal{X}^k\mathcal{Y}$ -cycle near  $x_i^\mathcal{X}$  to lie on the wrong side of  $\Sigma$ . If  $y_i$  and  $y_{i+n}$  both lie in  $\Sigma$ , for some  $i \geq 0$ , then our method of proof of admissibility fails because any fattening of  $\mathcal{L}(y_i, y_{i+n})$  into an open set (containing one point of each  $\mathcal{X}^k\mathcal{Y}$ -cycle, for sufficiently large  $k$ ) will not lie entirely on one side of  $\Sigma$ .
- As evident in the proof of Theorem 5.1, the assumption  $c > \lambda_2$  ensures that  $\mathcal{X}^k\mathcal{Y}$ -cycles are admissible whereas  $c < 1$  ensures that they are asymptotically stable.
- In Theorem 5.2, the assumptions  $c \neq -1$  and  $c \neq 0$  eliminate the possibility of alternate geometric structures that have codimension greater than three.
- Assumption (4) of Theorem 5.2 is used in our proof to show that  $y_\alpha \in \Sigma$  and to establish the given properties regarding  $\mathcal{X}^k\mathcal{Y}^{\bar{0}}$ -cycles.

## 6 Examples for the three-dimensional border-collision normal form

For piecewise-smooth maps, interactions between invariant sets and switching manifolds give rise to so-called discontinuity-induced bifurcations [17]. The simplest bifurcation of this type corresponds to the collision of a fixed point with a switching manifold. If the map is continuous and the derivatives of its components are bounded, at least locally, then the bifurcation is referred to as a border-collision bifurcation.

Diverse dynamics may be created in border-collision bifurcations, see [18] for a recent review. These dynamics are described by maps of the form (2.1). If certain non-degeneracy conditions are satisfied, then the map can be put into the border-collision normal form [29]. In three dimensions, this normal form can be written as

$$x_{i+1} = \begin{cases} \begin{bmatrix} \tau_L & 1 & 0 \\ -\sigma_L & 0 & 1 \\ \delta_L & 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mu, & e_1^\top x_i \leq 0, \\ \begin{bmatrix} \tau_R & 1 & 0 \\ -\sigma_R & 0 & 1 \\ \delta_R & 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mu, & e_1^\top x_i \geq 0, \end{cases} \quad (6.1)$$

where  $\tau_L, \sigma_L, \delta_L, \tau_R, \sigma_R, \delta_R \in \mathbb{R}$  and  $\mu \in \mathbb{R}$  is a parameter that controls the border-collision bifurcation which occurs at  $\mu = 0$ . Since the structure of the dynamics of (6.1) is independent of the magnitude of  $\mu$ , it is appropriate to set  $\mu = 1$  in order to study dynamics created in border-collision bifurcations. With this restriction, the parameter space of (6.1) is six-dimensional.

Given suitable words  $\mathcal{X}$  and  $\mathcal{Y}$ , we can search for codimension-three points in the parameter space of (6.1) at which the map has infinitely many asymptotically stable  $\mathcal{X}^k \mathcal{Y}$ -cycles. To do this, we first identify a computationally convenient set of three codimension-one conditions that describe the scenario of Theorems 5.1 and 5.2.

With a subsumed homoclinic connection,  $f_{\mathcal{Y}}$  maps  $E^u(x_0^{\mathcal{X}})$  onto  $E^s(x_0^{\mathcal{X}})$ , see Fig. 4. To describe this restriction algebraically, we consider the scalar quantity  $a = \omega_1^\top (x - x_0^{\mathcal{X}})$  which is a measure of the displacement of  $x \in \mathbb{R}^N$  from  $E^s(x_0^{\mathcal{X}})$ . Since  $f_{\mathcal{Y}}$  is affine, this displacement changes affinely under  $f_{\mathcal{Y}}$ . That is,  $\omega_1^\top (f_{\mathcal{Y}}(x) - x_0^{\mathcal{X}}) = \gamma_{11}a + \psi_1$ , for some constants  $\gamma_{11}, \psi_1 \in \mathbb{R}$  (using the notation of §7). Thus with a subsumed homoclinic connection we must have  $\gamma_{11} = \psi_1 = 0$ .

The scenario of Theorems 5.1 and 5.2 also requires  $\lambda_1 \lambda_2 = 1$ . The three examples given below were obtained by fixing the values of  $\sigma_L, \sigma_R$  and  $\delta_R$  and solving

$$\gamma_{11} = 0, \quad \psi_1 = 0, \quad \lambda_1 \lambda_2 = 1, \quad (6.2)$$

for  $\tau_L, \tau_R$  and  $\delta_L$ . This is an effective approach because each of the constants in (6.2) can be evaluated numerically in a relatively efficient manner at any point in the parameter space of (6.1). Once such a point is found, its validity can be verified by checking each of the conditions in Theorems 5.1 and 5.2.

First we consider

$$\mathcal{X} = RLLR, \quad \mathcal{Y} = LLR. \quad (6.3)$$

For these words, infinitely many asymptotically stable  $\mathcal{X}^k\mathcal{Y}$ -cycles were identified in [22] for the two-dimensional border-collision normal form. Using the approach described above, for the three-dimensional system (6.1) we obtained the values

$$\begin{aligned}\tau_L &= 1.1770635074, & \sigma_L &= 1, & \delta_L &= 0.4334058651, \\ \tau_R &= -1.0170722063, & \sigma_R &= 0.5, & \delta_R &= 1,\end{aligned}\tag{6.4}$$

given to ten decimal places.

Fig. 5 shows the first two coordinates of the phase space of (6.1) with (6.4). One branch of  $W^u(\{x_i^\mathcal{X}\})$  appears to converge to the  $\mathcal{X}$ -cycle, thus forming a subsumed homoclinic connection. Strictly speaking the homoclinic connection is not (quite) subsumed because the parameter values are only accurate to ten decimal places. The part of  $W^u(\{x_i^\mathcal{X}\})$  that emanates from  $x_0^\mathcal{X}$  converges to  $x_1^\mathcal{X}$ . This is because  $d = 1$ , where  $d$  is given by (3.4) with  $n = 4$  and  $p = 3$ . Also the  $\mathcal{S}$ -orbit  $\{y_i\}$  has  $y_2 \in \Sigma$  because  $\alpha = 2$ .

The  $\mathcal{X}^k\mathcal{Y}$ -cycles are admissible and asymptotically stable for at least  $k = 0, 1, \dots, 7$ , as shown in Fig. 5. The corresponding  $\mathcal{X}^k\mathcal{Y}^\bar{0}$ -cycles are also shown and approach the  $\mathcal{S}$ -orbit with increasing values of  $k$ , as predicted by Theorem 5.2. In order to distinguish the different periodic solutions clearly, we have connected the corresponding pairs of  $\mathcal{X}^k\mathcal{Y}$  and  $\mathcal{X}^k\mathcal{Y}^\bar{0}$ -cycles with line segments (these line segments do not relate to the dynamics of (6.1)).

Fig. 6 provides a three-dimensional view of phase space about  $x_0^\mathcal{X}$  for this example. Here we can observe several expected features of  $W^u(\{x_i^\mathcal{X}\})$ . The part of this manifold that converges to  $x_0^\mathcal{X}$  (with forward iterations) is comprised of a union of line segments (connecting  $y_3, y_7, y_{11}$ , etc). It lies within  $W^s(\{x_i^\mathcal{X}\})$  and approaches  $x_0^\mathcal{X}$  in the slow  $\zeta_2$  direction.

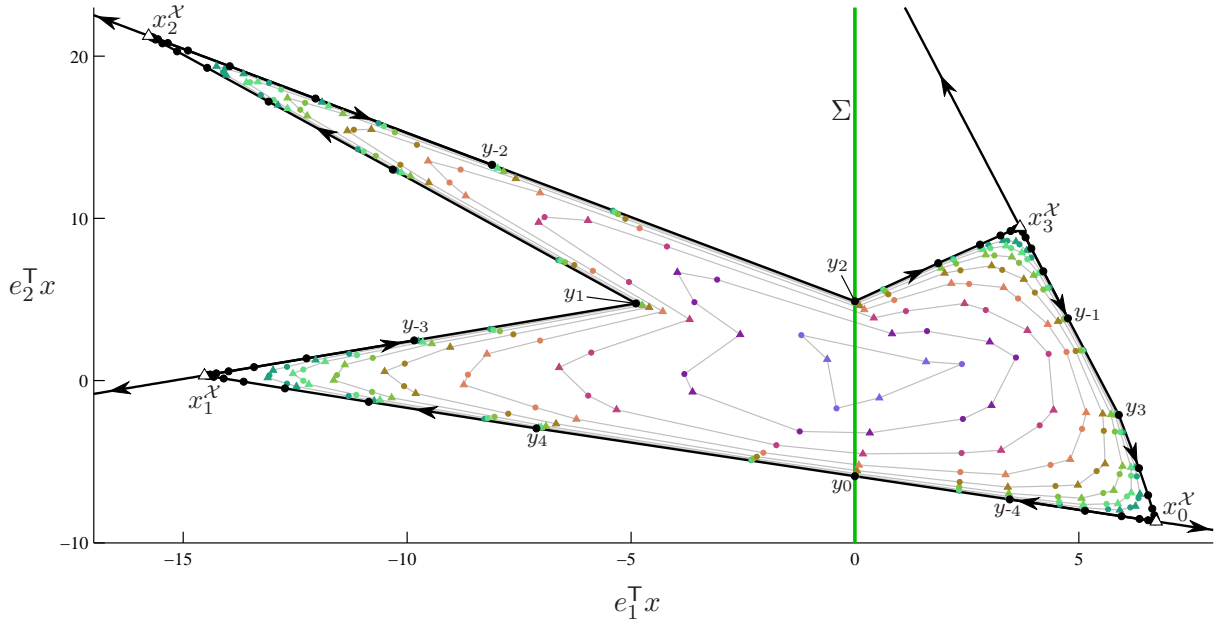


Figure 5: A phase portrait of (6.1) with (6.4). We show the  $\mathcal{X}$ -cycle and the homoclinic  $\mathcal{S}$ -orbit  $\{y_i\}$ , where  $\mathcal{X}$  and  $\mathcal{Y}$  are given by (6.3). We use circles to show asymptotically stable  $\mathcal{X}^k\mathcal{Y}$ -cycles for  $k = 0, \dots, 7$ , and triangles to show unstable  $\mathcal{X}^k\mathcal{Y}^\bar{0}$ -cycles for the same values of  $k$ .

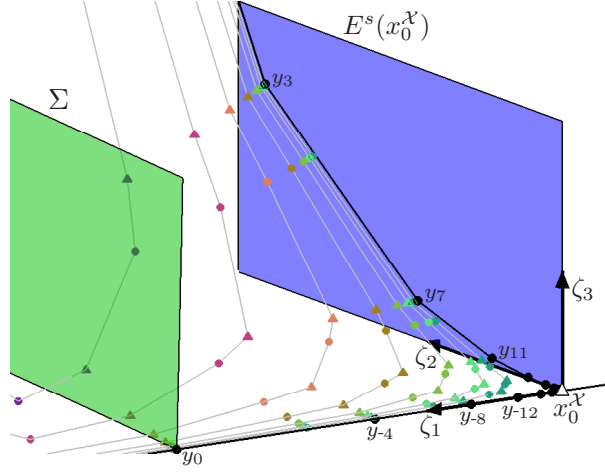


Figure 6: A three-dimensional view of the phase portrait of (6.1) with (6.4) (shown in Fig. 5) using eigenvectors of  $M_{\mathcal{X}}$  ( $\zeta_1$ ,  $\zeta_2$  and  $\zeta_3$ ) as coordinate axes.

Secondly we consider

$$\mathcal{X} = RLR, \quad \mathcal{Y} = LL, \quad (6.5)$$

in order to show that  $\mathcal{Y}$  need not consist of the last few symbols of  $\mathcal{X}$ . For (6.5) we obtained the values

$$\begin{aligned} \tau_L &= -1.9465556255, & \sigma_L &= -1, & \delta_L &= 0.3387541740, \\ \tau_R &= -0.3249411658, & \sigma_R &= 1, & \delta_R &= 0.9, \end{aligned} \quad (6.6)$$

and with these values a phase portrait of (6.1) is shown in Fig. 7. The words (6.5) satisfy (3.8) with  $\alpha = 4$ , and so  $y_4 \in \Sigma$ . Also  $d = 1$ , so part of the branch of  $W^u(\{x_i^{\mathcal{X}}\})$  that emanates from  $x_0^{\mathcal{X}}$  converges to  $x_1^{\mathcal{X}}$ . The other branch of  $W^u(\{x_i^{\mathcal{X}}\})$  converges to an asymptotically stable  $LLR$ -cycle. Asymptotically stable  $\mathcal{X}^k\mathcal{Y}$ -cycles and unstable  $\mathcal{X}^k\mathcal{Y}^{\bar{0}}$ -cycles are shown for  $k = 2, \dots, 9$  (for  $k = 0$  and  $k = 1$  they are not admissible).

Finally we consider

$$\mathcal{X} = RLRLRRLR, \quad \mathcal{Y} = LRRLR. \quad (6.7)$$

This example demonstrates that the scenario of Theorems 5.1 and 5.2 is not restricted to only the simplest choices of  $\mathcal{X}$  and  $\mathcal{Y}$ . For these words we obtained

$$\begin{aligned} \tau_L &= -0.5298581051, & \sigma_L &= 0.5, & \delta_L &= -0.2220122186, \\ \tau_R &= -3.4893057804, & \sigma_R &= 1.6, & \delta_R &= 0.6, \end{aligned} \quad (6.8)$$

with which (6.1) is not invertible because  $\delta_L\delta_R < 0$ . Here  $n = 8$  and  $p = 5$  and so by (3.4) we have  $d = 3$ . Also  $\alpha = 1$ . Fig. 8 shows a phase portrait of this example, including asymptotically stable  $\mathcal{X}^k\mathcal{Y}$ -cycles and unstable  $\mathcal{X}^k\mathcal{Y}^{\bar{0}}$ -cycles for  $k = 0, \dots, 7$ .

## 7 Proofs of the main results

In order to prove Theorems 5.1 and 5.2, we first introduce a coordinate system centred about  $x_0^{\mathcal{X}}$  and aligned with the eigenvectors  $\zeta_1$  and  $\zeta_2$  and describe some basic properties of the map  $g$  in



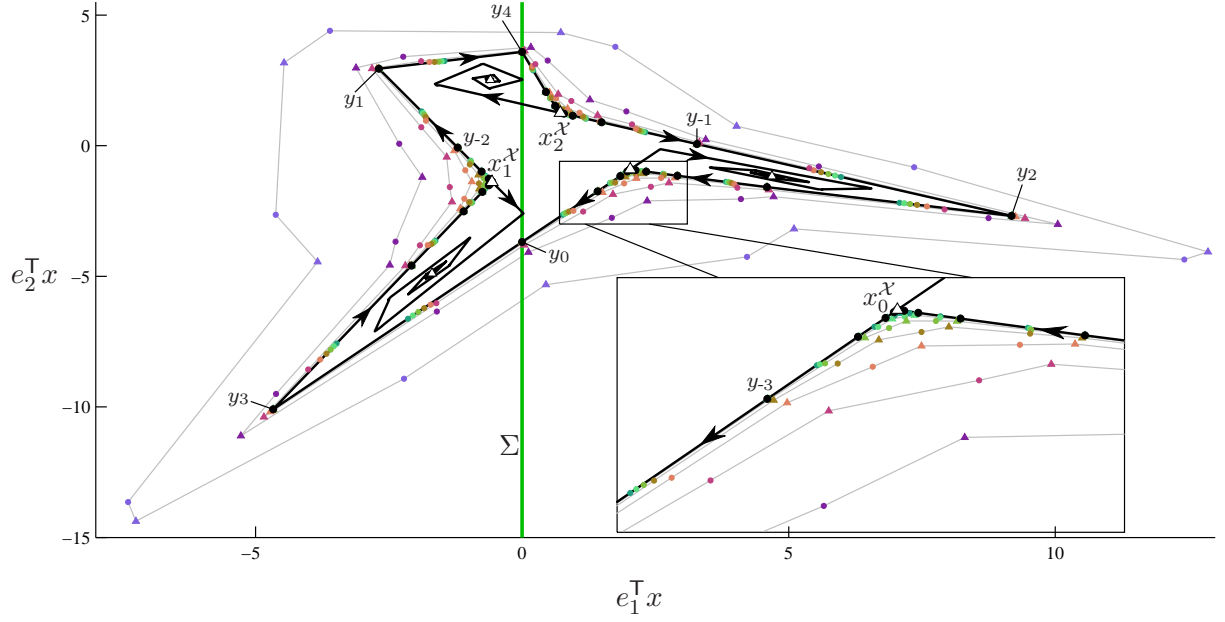


Figure 7: A phase portrait of (6.1) with (6.6) using the same conventions as Fig. 5. Here  $\mathcal{X}$  and  $\mathcal{Y}$  are given by (6.5) and the  $\mathcal{X}^k \mathcal{Y}$  and  $\mathcal{X}^k \mathcal{Y}^0$ -cycles are shown for  $k = 2, \dots, 9$ .

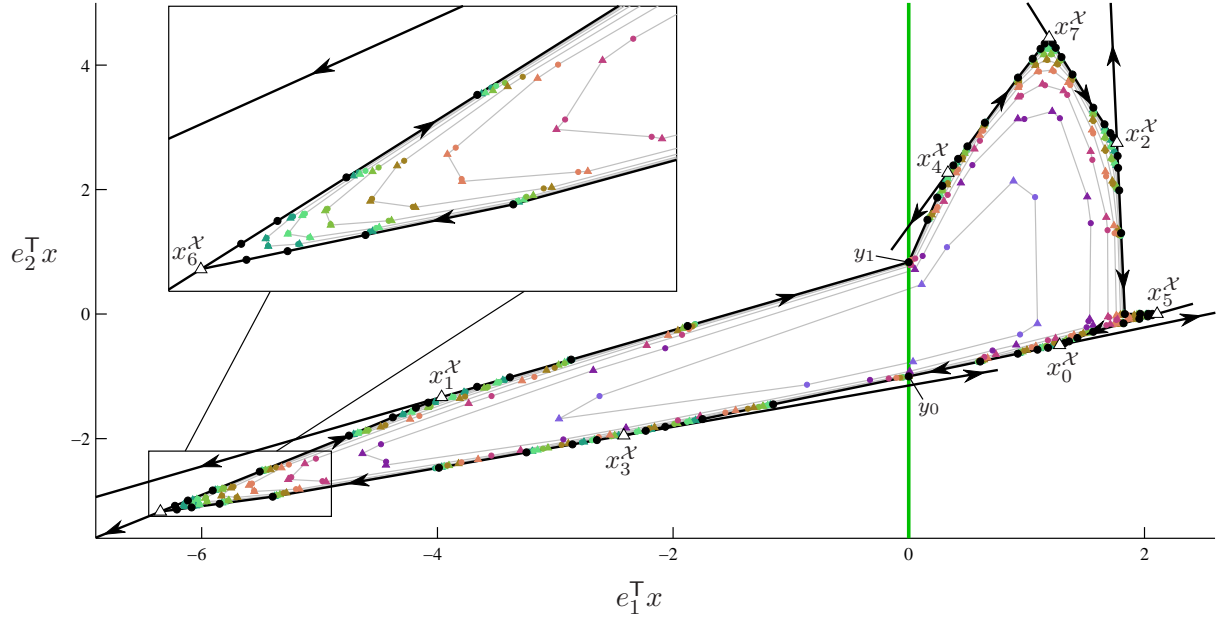


Figure 8: A phase portrait of (6.1) with (6.8) using the same conventions as Fig. 5. Here  $\mathcal{X}$  and  $\mathcal{Y}$  are given by (6.7) and the  $\mathcal{X}^k \mathcal{Y}$  and  $\mathcal{X}^k \mathcal{Y}^0$ -cycles are shown for  $k = 0, \dots, 7$ .

these coordinates. Here it is only necessary to assume that  $\mathcal{X}$  and  $\mathcal{Y}$  are words of length  $n$  and  $p$  and that the eigenvalues of  $M_{\mathcal{X}}$  satisfy (4.1).



Let  $J$  be a matrix similar to  $M_{\mathcal{X}}$  and with the block diagonal form

$$J = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \tilde{J} \end{bmatrix}, \quad (7.1)$$

where  $\tilde{J}$  is an  $(N-2) \times (N-2)$  matrix. We could take  $J$  to be a real Jordan form of  $M_{\mathcal{X}}$ , but need not be so precise as for our purposes we only require that large powers of  $J$  can be written asymptotically as

$$J^k = \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \mathcal{O}(\beta^k) \end{bmatrix}, \quad (7.2)$$

where  $\mathcal{O}$  represents big-O notation. There exists a real-valued non-singular matrix  $Q$  such that

$$J = Q^{-1} M_{\mathcal{X}} Q,$$

and

$$\zeta_1 = Q e_1, \quad \zeta_2 = Q e_2. \quad (7.3)$$

In view of (4.2),

$$\omega_1^\top = e_1^\top Q^{-1}, \quad \omega_2^\top = e_2^\top Q^{-1}. \quad (7.4)$$

We then let

$$u = h(x) = Q^{-1}(x - x_0^{\mathcal{X}}).$$

These “ $u$ -coordinates” are centred about  $x_0^{\mathcal{X}}$  and partially align with the invariant subspaces of  $x_0^{\mathcal{X}}$  for the map  $f_{\mathcal{X}}$ . In particular,  $x \in E^u(x_0^{\mathcal{X}})$  if and only if  $h(x) = a e_1$  for some  $a \in \mathbb{R}$ , and  $x \in E^s(x_0^{\mathcal{X}})$  if and only if  $e_1^\top h(x) = 0$ .

We let  $g = h \circ f \circ h^{-1}$  and  $g_{\mathcal{Z}} = h \circ f_{\mathcal{Z}} \circ h^{-1}$  denote the transformations of  $f$  and  $f_{\mathcal{Z}}$  to  $u$ -coordinates, for any word  $\mathcal{Z}$ . In particular,

$$g_{\mathcal{X}}(u) = J u, \quad (7.5)$$

and

$$g_{\mathcal{Y}}(u) = \Gamma u + \psi, \quad (7.6)$$

where

$$\Gamma = Q^{-1} M_{\mathcal{Y}} Q, \quad \psi = Q^{-1} (P_{\mathcal{Y}} b - (I - M_{\mathcal{Y}}) x_0^{\mathcal{X}}).$$

For each  $i, j = 1, \dots, N$ , we let  $\gamma_{ij}$  denote the  $(i, j)$ -element of  $\Gamma$  and  $\psi_i = e_i^\top \psi$ . Then by (7.3) and (7.4) the quantity (5.1) is given by

$$c = \det \left( \begin{bmatrix} e_1^\top \\ e_2^\top \end{bmatrix} \Gamma \begin{bmatrix} e_1 & e_2 \end{bmatrix} \right) = \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}. \quad (7.7)$$

Given any  $u \in \mathbb{R}^N$  and any word  $\mathcal{Z}$ , we say that  $u$  follows  $\mathcal{Z}$  under  $g$  if  $h^{-1}(u)$  follows  $\mathcal{Z}$  under  $f$ .

We first provide the following algebraic result that is used below in the proofs of both Theorem 5.1 and Theorem 5.2. This result is useful because in both proofs  $y_0 = h^{-1}(a_0 e_1)$  follows both  $\mathcal{X}\mathcal{Y}$  and  $\mathcal{Y}\mathcal{X}$  under  $f$ .

**Lemma 7.1.** *If  $a_0 \in \mathbb{R}$  is such that the point  $a_0 e_1$  follows both  $\mathcal{X}\mathcal{Y}$  and  $\mathcal{Y}\mathcal{X}$  under  $g$ , then*

$$\psi_2 = \frac{-a_0(\lambda_1 - \lambda_2)\gamma_{21}}{1 - \lambda_2}. \quad (7.8)$$

*Proof.* Since  $a_0 e_1$  follows  $\mathcal{X}\mathcal{Y}$  under  $g$ , we have  $g^{n+p}(a_0 e_1) = g_{\mathcal{Y}}(g_{\mathcal{X}}(a_0 e_1))$ . Thus by (7.5) and (7.6),

$$g^{n+p}(a_0 e_1) = a_0 \Gamma J e_1 + \psi. \quad (7.9)$$

Similarly, since  $a_0 e_1$  follows  $\mathcal{Y}\mathcal{X}$  under  $g$ , we have  $g^{n+p}(a_0 e_1) = g_{\mathcal{X}}(g_{\mathcal{Y}}(a_0 e_1))$  and thus

$$g^{n+p}(a_0 e_1) = a_0 J \Gamma e_1 + J \psi. \quad (7.10)$$

By matching the second components of (7.9) and (7.10) we obtain

$$a_0 \lambda_2 \gamma_{21} + \lambda_2 \psi_2 = a_0 \lambda_1 \gamma_{21} + \psi_2,$$

where we have used (7.1). By solving this equation for  $\psi_2$  we obtain (7.8).  $\square$

## 7.1 Proof of Theorem 5.1

**Step 1.** Here we establish some important properties of the homoclinic  $\mathcal{S}$ -orbit  $\{y_i\}$  and describe some of its points in  $u$ -coordinates.

By assumption,  $y_0$  follows  $\mathcal{Y}\mathcal{X}^\infty$  under  $f$ . As noted in the proof of Proposition 4.1, since  $\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\overline{0\alpha}}$  and  $y_0, y_\alpha \in \Sigma$ , by Lemma 2.1  $y_0$  also follows  $\mathcal{X}\mathcal{Y}\mathcal{X}^\infty$  under  $f$ . Thus for all  $j \leq 0$ , the point  $y_{j_n}$  follows  $\mathcal{X}$  under  $f$ .

Since  $y_0 \in E^u(x_0^\mathcal{X})$  by definition, and  $y_{j_n} \rightarrow x_0^\mathcal{X}$  as  $j \rightarrow -\infty$ , we conclude that  $y_{j_n} \in E^u(x_0^\mathcal{X})$  for all  $j \leq 1$ . In addition, we can write

$$h(y_{j_n}) = a_j e_1, \quad \text{for all } j \leq 1, \quad (7.11)$$

where

$$a_j = a_0 \lambda_1^j, \quad (7.12)$$

and  $a_0 \neq 0$  because  $y_0 \in \Sigma$  and  $x_0^\mathcal{X} \notin \Sigma$  and so  $y_0 \neq x_0^\mathcal{X}$ .

For each  $i = 0, \dots, n-1$  and all  $j \leq -1$ , we have  $y_{j_n+i} \in \mathcal{L}(x_i^\mathcal{X}, y_i)$ . Since  $y_0$  follows  $\mathcal{X}$  under  $f$ , each  $y_i$  (for  $i = 0, \dots, n-1$ ) follows  $\mathcal{X}_i$  under  $f$ . Thus since the  $\mathcal{S}$ -orbit and the  $\mathcal{X}$ -cycle are admissible, each  $y_i$  either lies on  $\Sigma$  or on the same side of  $\Sigma$  as  $x_i^\mathcal{X}$ . Since each  $x_i^\mathcal{X} \notin \Sigma$  by assumption (4), we have  $\mathcal{L}(x_i^\mathcal{X}, y_i) \cap \Sigma = \emptyset$  and thus

$$y_{j_n+i} \notin \Sigma, \quad \text{for each } i = 0, \dots, n-1 \text{ and all } j \leq -1. \quad (7.13)$$

**Step 2.** Here we show that every  $x \in E^u(x_0^\mathcal{X})$  maps to  $E^s(x_0^\mathcal{X})$  under  $f_{\mathcal{Y}}$ . In terms of  $g_{\mathcal{Y}}$ , this equates to  $\gamma_{11} = \psi_1 = 0$ .

Since  $y_0$  follows  $\mathcal{Y}\mathcal{X}^\infty$  under  $f$  (by definition) and  $\{y_i\}$  converges to the  $\mathcal{X}$ -cycle as  $i \rightarrow \infty$ , we must have  $f_{\mathcal{Y}}(y_0) \in E^s(x_0^\mathcal{X})$ . In Step 1 we showed that  $y_0$  also follows  $\mathcal{X}\mathcal{Y}\mathcal{X}^\infty$  under  $f$ , thus  $y_n$  follows  $\mathcal{Y}\mathcal{X}^\infty$  and so  $f_{\mathcal{Y}}(y_n) \in E^s(x_0^\mathcal{X})$ .

In terms of  $g_{\mathcal{Y}}$  we have  $e_1^\top g_{\mathcal{Y}}(a_0 e_1) = 0$  and  $e_1^\top g_{\mathcal{Y}}(a_1 e_1) = 0$ . Since  $a_0 \neq a_1$  and  $e_1^\top g_{\mathcal{Y}}(a e_1) = \gamma_{11}a + \psi_1$ , where  $a \in \mathbb{R}$ , we must have

$$\gamma_{11} = 0, \quad \psi_1 = 0. \quad (7.14)$$

That is, every  $x \in E^u(x_0^{\mathcal{X}})$  maps to  $E^s(x_0^{\mathcal{X}})$  under  $f_{\mathcal{Y}}$ . Also

$$c = -\gamma_{12}\gamma_{21}, \quad (7.15)$$

by (7.7).

**Step 3.** Next we use the previous result to show that  $\det(I - M_{\mathcal{X}^k \mathcal{Y}}) \neq 0$  and thus  $\mathcal{X}^k \mathcal{Y}$ -cycles exist and are unique, for sufficiently large values of  $k$ .

By (2.6),  $M_{\mathcal{X}^k \mathcal{Y}} = M_{\mathcal{Y}} M_{\mathcal{X}}^k$ . Also  $\Gamma J^k = Q^{-1} M_{\mathcal{Y}} M_{\mathcal{X}}^k Q$ , thus  $\det(I - M_{\mathcal{X}^k \mathcal{Y}}) = \det(I - \Gamma J^k)$ . In view of (7.14) and the assumption  $\lambda_1 \lambda_2 = 1$ ,

$$\Gamma J^k = \begin{bmatrix} 0 & \frac{\gamma_{12}}{\lambda_1^k} \\ \gamma_{21} \lambda_1^k & \frac{\gamma_{22}}{\lambda_1^k} \\ \vdots & \vdots \\ \gamma_{N1} \lambda_1^k & \frac{\gamma_{N2}}{\lambda_1^k} \end{bmatrix} \mathcal{O}(\beta^k), \quad (7.16)$$

and thus

$$\det(I - \Gamma J^k) = 1 + c - \frac{\gamma_{22}}{\lambda_1^k} + \mathcal{O}(\lambda_1^k \beta^k), \quad (7.17)$$

where we have also used (7.15). The leading order terms in (7.17) stem from the top-left  $2 \times 2$  block of  $\Gamma J^k$ .

Notice  $\lambda_1 \beta < 1$  because  $\beta < \lambda_2$  and  $\lambda_1 \lambda_2 = 1$ . Thus  $\det(I - \Gamma J^k) \rightarrow 1 + c$  as  $k \rightarrow \infty$ . Since  $c \neq -1$  by assumption (3), we have  $\det(I - \Gamma J^k) \neq 0$  and thus a unique  $\mathcal{X}^k \mathcal{Y}$ -cycle, that we denote  $\{x_i^{\mathcal{X}^k \mathcal{Y}}\}$ , for sufficiently large values of  $k$ .

**Step 4.** Here we derive asymptotic expressions for the points of the  $\mathcal{X}^k \mathcal{Y}$ -cycles associated with  $x_0^{\mathcal{X}}$  in  $u$ -coordinates.

The point  $x_0^{\mathcal{X}^k \mathcal{Y}}$  is a fixed point of  $f_{\mathcal{Y}} \circ f_{\mathcal{X}}^k$ . Thus  $h(x_0^{\mathcal{X}^k \mathcal{Y}})$  is a fixed point of  $g_{\mathcal{Y}} \circ g_{\mathcal{X}}^k$ . By (7.5) and (7.6),

$$h(x_0^{\mathcal{X}^k \mathcal{Y}}) = (I - \Gamma J^k)^{-1} \psi. \quad (7.18)$$

In view of the definition of an adjugate matrix (as the transpose of a cofactor matrix), we evaluate the minors of  $I - \Gamma J^k$  using (7.16) to produce

$$\text{adj}(I - \Gamma J^k) \psi = \begin{bmatrix} \frac{\gamma_{12}\psi_2}{\lambda_1^k} + \mathcal{O}(\beta^k) \\ \psi_2 + \mathcal{O}(\lambda_1^k \beta^k) \\ \mathcal{O}(1) \\ \vdots \\ \mathcal{O}(1) \end{bmatrix}. \quad (7.19)$$

By then using (7.17) and (7.19) to evaluate (7.18) we obtain

$$h(x_0^{\mathcal{X}^k \mathcal{Y}}) = \begin{bmatrix} \frac{\gamma_{12}\psi_2}{(1+c)\lambda_1^k} + \mathcal{O}\left(\frac{1}{\lambda_1^{2k}}\right) + \mathcal{O}(\beta^k) \\ \frac{\psi_2}{1+c} + \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) + \mathcal{O}(\lambda_1^k \beta^k) \\ \mathcal{O}(1) \\ \vdots \\ \mathcal{O}(1) \end{bmatrix}. \quad (7.20)$$

For all  $j = 1, \dots, k$ , we have  $x_{jn}^{\mathcal{X}^k \mathcal{Y}} = f_{\mathcal{X}}^j(x_0^{\mathcal{X}^k \mathcal{Y}})$ . Therefore, by (7.5),

$$h(x_{jn}^{\mathcal{X}^k \mathcal{Y}}) = J^j h(x_0^{\mathcal{X}^k \mathcal{Y}}), \quad \text{for all } j = 0, \dots, k. \quad (7.21)$$

In particular, by (7.20) and (7.21) we can write

$$h(x_{kn}^{\mathcal{X}^k \mathcal{Y}}) = \frac{\gamma_{12}\psi_2}{1+c} e_1 + \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) + \mathcal{O}(\lambda_1^k \beta^k). \quad (7.22)$$

**Step 5.** Next we identify a neighbourhood of  $x_0^{\mathcal{X}}$  within which all points follow  $\mathcal{X}$  under  $f$ . As a first step to demonstrating the admissibility of the  $\mathcal{X}^k \mathcal{Y}$ -cycles, we use (7.21) to show that, for sufficiently large values of  $k$ , all but a handful of the  $x_{jn}^{\mathcal{X}^k \mathcal{Y}}$  (which are dealt with in Step 6) lie within this neighbourhood.

By assumption (4), the  $\mathcal{X}$ -cycle is admissible with no points on  $\Sigma$ . Thus there exists a neighbourhood of  $x_0^{\mathcal{X}}$  within which all points follow  $\mathcal{X}$  under  $f$ . In  $u$ -coordinates this means that there exists  $r > 0$  such that any point in the open ball

$$B_r = \{u \in \mathbb{R}^N \mid \|u\| < r\} \quad (7.23)$$

follows  $\mathcal{X}$  under  $g$ . By (7.20) and (7.21), there exists  $j^* \in \mathbb{Z}$  such that for all  $k > 2j^*$  and all  $j^* \leq j < k - j^*$ , we have  $h(x_{jn}^{\mathcal{X}^k \mathcal{Y}}) \in B_r$ , see Fig. 9. That is,  $h(x_{jn}^{\mathcal{X}^k \mathcal{Y}})$  follows  $\mathcal{X}$  under  $g$  for all  $j^* \leq j < k - j^*$ . More succinctly,  $h(x_{j^*n}^{\mathcal{X}^k \mathcal{Y}})$  follows  $\mathcal{X}^{k-2j^*}$  under  $g$ .

**Step 6.** Here we complete the demonstration that  $\mathcal{X}^k \mathcal{Y}$ -cycles are admissible for sufficiently large values of  $k$ . In view of the final statement of Step 5, it remains to show that  $h(x_{(k-j^*)n}^{\mathcal{X}^k \mathcal{Y}})$  follows  $\mathcal{X}^{j^*} \mathcal{Y} \mathcal{X}^{j^*}$  under  $g$  for sufficiently large values of  $k$ . We do this by showing that  $h(x_{(k-j^*)n}^{\mathcal{X}^k \mathcal{Y}})$  approaches the line segment

$$L_{j^*} = \mathcal{L}(h(y_{-j^*n}), h(y_{(-j^*+1)n})) , \quad (7.24)$$

as  $k \rightarrow \infty$ , see Fig. 9.

First, recall that  $y_0$  follows  $\mathcal{Y} \mathcal{X}^\infty$  under  $f$  and thus  $y_{-j^*n}$  follows  $\mathcal{X}^{j^*} \mathcal{Y} \mathcal{X}^\infty$  under  $f$ . As shown in Step 1,  $y_0$  follows  $\mathcal{X} \mathcal{Y} \mathcal{X}^\infty$  under  $f$ , thus  $y_{(-j^*+1)n}$  also follows  $\mathcal{X}^{j^*} \mathcal{Y} \mathcal{X}^\infty$  under  $f$ . By Lemma 2.2 every point in  $\mathcal{L}(y_{-j^*n}, y_{(-j^*+1)n})$  follows  $\mathcal{X}^{j^*} \mathcal{Y} \mathcal{X}^\infty$  under  $f$ . In terms of  $u$ -coordinates, this says that every  $u \in L_{j^*}$  follows  $\mathcal{X}^{j^*} \mathcal{Y} \mathcal{X}^\infty$  under  $g$ .

By assumption (6) and (7.13), the line segment  $f^i(\mathcal{L}(y_{-j^*n}, y_{(-j^*+1)n}))$  does not have both endpoints in  $\Sigma$  for any  $i \geq 0$ . Thus  $f^i(\mathcal{L}(y_{-j^*n}, y_{(-j^*+1)n})) \cap \Sigma = \emptyset$ , for all  $i \geq 0$ . Translated into  $u$ -coordinates, this says

$$g^i(L_{j^*}) \cap h(\Sigma) = \emptyset, \quad \text{for all } i \geq 0. \quad (7.25)$$

By (7.11), we can write

$$L_{j^*} = \{ae_1 \mid a_{-j^*} < a < a_{-j^*+1}\}. \quad (7.26)$$

We now fatten  $L_{j^*}$  into a rugby ball shaped region

$$\Omega_{j^*, \kappa} = \{u \in \mathbb{R}^N \mid a = e_1^\top u, a_{-j^*} < a < a_{-j^*+1}, \|u - ae_1\| < \kappa(a - a_{-j^*})(a_{-j^*+1} - a)\}, \quad (7.27)$$

where  $\kappa > 0$  is a measure of the magnitude of fattening, see Fig. 9. By (7.25) there exists  $\kappa > 0$  such that  $g^i(\Omega_{j^*, \kappa}) \cap h(\Sigma) = \emptyset$  for all  $i = 0, \dots, 2j^*n + p - 1$ . This is because each  $g^i(L_{j^*})$  is either bounded away from  $h(\Sigma)$  or protrudes from  $h(\Sigma)$  at a non-zero angle, so we can find  $\kappa > 0$  such that the finite collection of fattened sets  $\{g^i(\Omega_{j^*, \kappa})\}_{i=0}^{2j^*n+p-1}$  does not intersect  $h(\Sigma)$ . Therefore every  $u \in \Omega_{j^*, \kappa}$  follows  $\mathcal{X}^{j^*}\mathcal{Y}\mathcal{X}^{j^*}$  under  $g$ .

We now fix  $\kappa > 0$  at such a value. It remains to show that  $h(x_{(k-j^*)n}^{\mathcal{X}^k\mathcal{Y}}) \in \Omega_{j^*, \kappa}$  for sufficiently large values of  $k$ . To do this we substitute (7.8) (which is valid because  $a_0e_1 = h(y_0)$  follows both  $\mathcal{X}\mathcal{Y}$  and  $\mathcal{Y}\mathcal{X}$  under  $g$ ) with  $\lambda_1\lambda_2 = 1$  into (7.22) to obtain

$$h(x_{kn}^{\mathcal{X}^k\mathcal{Y}}) = \frac{a_0(\lambda_1 + 1)c}{1 + c} e_1 + \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) + \mathcal{O}(\lambda_1^k \beta^k), \quad (7.28)$$

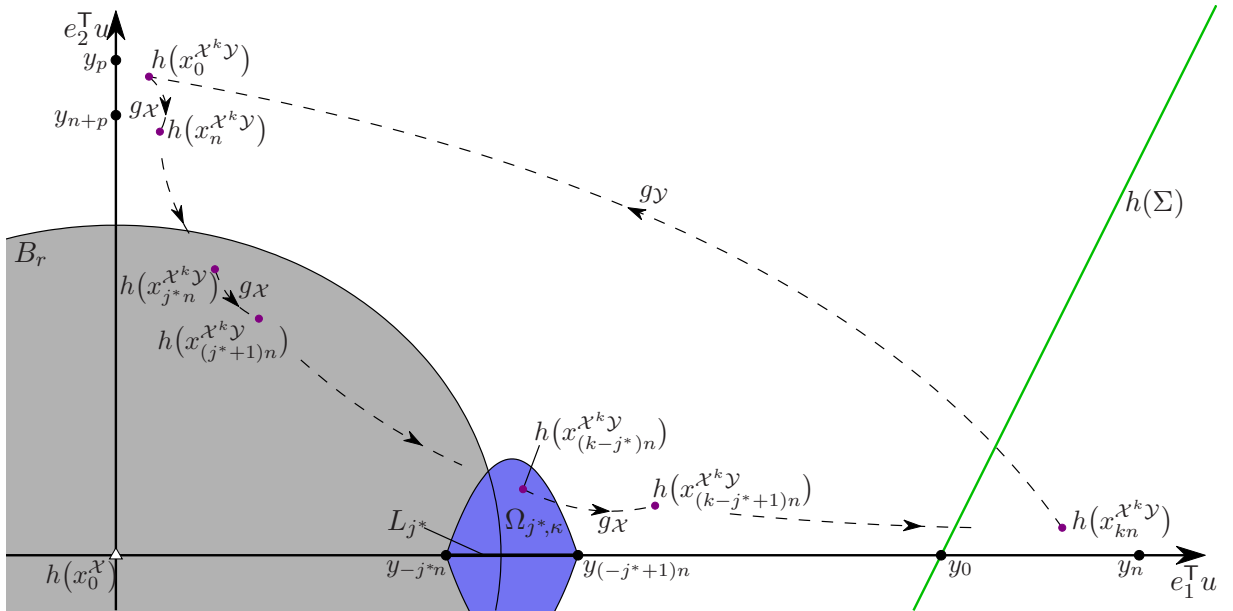


Figure 9: A sketch of the phase space of  $f$  in  $u$ -coordinates (7.5). The first axis  $e_1^\top u$  contains  $E^u(x_0^{\mathcal{X}})$ . The second axis  $e_2^\top u$  corresponds to the slowest direction within  $E^s(x_0^{\mathcal{X}})$ . We show some points of the homoclinic  $\mathcal{S}$ -orbit  $\{y_i\}$  and a typical  $\mathcal{X}^k\mathcal{Y}$ -cycle.

where we have also used (7.15). Thus by (7.12) and (7.21),

$$h\left(x_{(k-j^*)n}^{\mathcal{X}^k\mathcal{Y}}\right) = \frac{a_{-j^*}(\lambda_1 + 1)c}{1 + c} e_1 + \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) + \mathcal{O}(\lambda_1^k \beta^k). \quad (7.29)$$

The assumption  $\lambda_2 < c < 1$  implies

$$1 < \frac{(\lambda_1 + 1)c}{1 + c} < \lambda_1, \quad (7.30)$$

and thus

$$a_{-j^*} < \lim_{k \rightarrow \infty} e_1^\top h\left(x_{(k-j^*)n}^{\mathcal{X}^k\mathcal{Y}}\right) < a_{-j^*+1}. \quad (7.31)$$

Hence  $h\left(x_{(k-j^*)n}^{\mathcal{X}^k\mathcal{Y}}\right) \in \Omega_{j^*, \kappa}$  and so the  $\mathcal{X}^k\mathcal{Y}$ -cycle is admissible, for sufficiently large values of  $k$ .

**Step 7.** Finally we verify the asymptotic stability of the  $\mathcal{X}^k\mathcal{Y}$ -cycles by showing that all eigenvalues of  $\Gamma J^k$ , and hence also of  $M_{\mathcal{X}^k\mathcal{Y}}$ , have modulus less than 1 for sufficiently large values of  $k$ . To do this we apply the Gershgorin circle theorem to a matrix similar to  $\Gamma J^k$ .

By (7.15) and (7.16), it can be seen that  $\Gamma J^k$  has an eigenvalue that converges to the purely imaginary value  $i\sqrt{c}$  as  $k \rightarrow \infty$ . The corresponding eigenvector is asymptotic to

$$v = \begin{bmatrix} \frac{i\sqrt{c}}{\lambda_1^k} \\ \gamma_{21} \\ \vdots \\ \gamma_{N1} \end{bmatrix}. \quad (7.32)$$

This motivates use of the non-singular matrix

$$T = \begin{bmatrix} v & \bar{v} & \begin{bmatrix} O \\ I \end{bmatrix} \end{bmatrix}, \quad (7.33)$$

where  $\bar{v}$  is the complex conjugate of  $v$ ,  $O$  is a  $2 \times (N-2)$  block of zeros, and  $I$  is the  $(N-2) \times (N-2)$  identity matrix. Direct calculations produce

$$T^{-1} \Gamma J^k T = \begin{bmatrix} i\sqrt{c} + \mathcal{O}(\lambda_1^k \beta^k) & \mathcal{O}(\lambda_1^k \beta^k) & \mathcal{O}(\lambda_1^k \beta^k) \\ \mathcal{O}(\lambda_1^k \beta^k) & -i\sqrt{c} + \mathcal{O}(\lambda_1^k \beta^k) & \mathcal{O}(\lambda_1^k \beta^k) \\ \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) & \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) & \mathcal{O}(\beta^k) \\ \vdots & \vdots & \\ \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) & \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) & \end{bmatrix}. \quad (7.34)$$

By the Gershgorin circle theorem, each eigenvalue of this matrix lies within  $\mathcal{O}(\lambda_1^k \beta^k)$  of  $i\sqrt{c}$  or  $-i\sqrt{c}$  or within  $\mathcal{O}\left(\frac{1}{\lambda_1^k}\right)$  of 0. Since eigenvalues are invariant under similarity transforms, the same is true for  $\Gamma J^k$ . Since  $\lambda_2 < c < 1$ , by assumption, all eigenvalues of  $\Gamma J^k$  therefore have modulus less than 1 for sufficiently large values of  $k$ .  $\blacksquare$

## 7.2 Proof of Theorem 5.2

**Step 1.** First we use the assumption that  $\mathcal{X}^k\mathcal{Y}$ -cycles are stable to obtain some restrictions on the coefficients describing  $f$ , in particular  $\gamma_{11} = 0$ .

If an  $\mathcal{X}^k\mathcal{Y}$ -cycle is stable then the eigenvalues of  $M_{\mathcal{X}^k\mathcal{Y}}$  have modulus less than or equal to 1. This implies  $|\text{trace}(M_{\mathcal{X}^k\mathcal{Y}})| \leq N$ . We have  $g_{\mathcal{X}^k\mathcal{Y}}(u) = \Gamma J^k u + \psi$ , from which we obtain

$$\text{trace}(M_{\mathcal{X}^k\mathcal{Y}}) = \text{trace}(\Gamma J^k) = \gamma_{11}\lambda_1^k + \gamma_{22}\lambda_2^k + \mathcal{O}(\beta^k). \quad (7.35)$$

Since  $\lambda_1 > 1$ , we must have

$$\gamma_{11} = 0, \quad (7.36)$$

and therefore by (7.7)

$$c = -\gamma_{12}\gamma_{21}. \quad (7.37)$$

By using (7.2) and substituting (7.36) and (7.37) we obtain

$$\det(I - \Gamma J^k) = 1 + c\lambda_1^k\lambda_2^k - \gamma_{22}\lambda_2^k + \mathcal{O}(\lambda_1^k\beta^k). \quad (7.38)$$

Again the assumption that  $\mathcal{X}^k\mathcal{Y}$ -cycles are stable implies  $\det(I - \Gamma J^k) \not\rightarrow \infty$  as  $k \rightarrow \infty$ . Since  $c \neq 0$  by assumption, we must have

$$\lambda_1\lambda_2 \leq 1. \quad (7.39)$$

If  $\lambda_1\lambda_2 = 1$ , then  $\det(I - \Gamma J^k) \rightarrow 1 + c$  as  $k \rightarrow \infty$ , while if  $0 < \lambda_1\lambda_2 < 1$  (which turns out not to be the case), then  $\det(I - \Gamma J^k) \rightarrow 1$  as  $k \rightarrow \infty$ . In either case the limiting value is nonzero because  $c \neq -1$  by assumption.

**Step 2.** Here we derive asymptotic expressions for points of the  $\mathcal{X}^k\mathcal{Y}$ -cycles associated with  $x_0^{\mathcal{X}}$  in  $u$ -coordinates.

We have  $h(x_0^{\mathcal{X}^k\mathcal{Y}}) = (I - \Gamma J^k)^{-1} \psi$  where  $\psi$  is not the zero vector (for otherwise  $h(x_0^{\mathcal{X}^k\mathcal{Y}}) = h(x_0^{\mathcal{X}})$  for all  $k$ , which is not possible). In addition,  $h(x_{jn}^{\mathcal{X}^k\mathcal{Y}}) = J^j h(x_0^{\mathcal{X}^k\mathcal{Y}})$  for all  $j = 0, \dots, k$ . By evaluating the minors of  $I - \Gamma J^k$  subject to (7.36) (in a similar fashion to Step 4 of the proof of Theorem 5.1), we obtain

$$h(x_{jn}^{\mathcal{X}^k\mathcal{Y}}) = \frac{1}{\det(I - \Gamma J^k)} \begin{bmatrix} \lambda_1^j (1 - \gamma_{22}\lambda_2^k) \psi_1 + \gamma_{12}\lambda_1^j\lambda_2^k\psi_2 + \mathcal{O}(\lambda_1^j\beta^k) \\ \gamma_{21}\lambda_1^k\lambda_2^j\psi_1 + \lambda_2^j\psi_2 + \mathcal{O}(\lambda_1^k\lambda_2^j\beta^k) \\ \mathcal{O}(\lambda_1^k\beta^j) \\ \vdots \\ \mathcal{O}(\lambda_1^k\beta^j) \end{bmatrix}, \quad \text{for all } j = 0, \dots, k. \quad (7.40)$$

**Step 3.** Here we use the admissibility of the  $\mathcal{X}^k\mathcal{Y}$ -cycles to obtain further restrictions on the coefficients describing  $f$ , in particular  $\psi_1 = 0$  and  $\lambda_1\lambda_2 = 1$ .

For each value of  $k$  for which the  $\mathcal{X}^k\mathcal{Y}$ -cycle is admissible with no points on  $\Sigma$ , the points  $x_{(k-1)n}^{\mathcal{X}^k\mathcal{Y}}$  and  $x_{kn}^{\mathcal{X}^k\mathcal{Y}}$  lie on different sides of  $\Sigma$  because  $\mathcal{X}_0 \neq \mathcal{Y}_0$ . Thus  $h(x_{(k-1)n}^{\mathcal{X}^k\mathcal{Y}})$  and  $h(x_{kn}^{\mathcal{X}^k\mathcal{Y}})$  lie on different sides of  $h(\Sigma)$  for arbitrarily large values of  $k$ .

But if  $\psi_1 \neq 0$ , then by (7.40) as  $k \rightarrow \infty$  the first components of  $h(x_{(k-1)n}^{\mathcal{X}^k\mathcal{Y}})$  and  $h(x_{kn}^{\mathcal{X}^k\mathcal{Y}})$  diverge while all other components converge. Thus  $h(\Sigma)$  must be parallel to the first coordinate

axis. In  $x$ -coordinates this says that  $\zeta_1$  is parallel to  $\Sigma$ . But this is not possible because  $e_1^\top \zeta_1 \neq 0$ , by assumption, thus

$$\psi_1 = 0. \quad (7.41)$$

Therefore (7.40) reduces to

$$h(x_{j_n}^{\mathcal{X}^k \mathcal{Y}}) = \frac{1}{\det(I - \Gamma J^k)} \begin{bmatrix} \gamma_{12} \lambda_1^j \lambda_2^k \psi_2 + \mathcal{O}(\lambda_1^j \beta^k) \\ \lambda_2^j \psi_2 + \mathcal{O}(\lambda_1^k \lambda_2^j \beta^k) \\ \mathcal{O}(\beta^j) \\ \vdots \\ \mathcal{O}(\beta^j) \end{bmatrix}, \quad \text{for all } j = 0, \dots, k. \quad (7.42)$$

Notice  $\gamma_{12} \neq 0$ , by (7.37) and because  $c \neq 0$  by assumption. Now suppose for a contradiction that either  $\lambda_1 \lambda_2 < 1$  or  $\psi_2 = 0$ . In this case  $h(x_{(k-1)n}^{\mathcal{X}^k \mathcal{Y}})$  and  $h(x_{kn}^{\mathcal{X}^k \mathcal{Y}})$  converge to the origin as  $k \rightarrow \infty$ . Since  $f$  is continuous, this implies that the distance between the  $(n+p)^{\text{th}}$  iterates of  $h(x_{(k-1)n}^{\mathcal{X}^k \mathcal{Y}})$  and  $h(x_{kn}^{\mathcal{X}^k \mathcal{Y}})$  tends to zero as  $k \rightarrow \infty$ . However, these iterates are the points  $h(x_0^{\mathcal{X}^k \mathcal{Y}})$  and  $h(x_n^{\mathcal{X}^k \mathcal{Y}})$ . We have  $h(x_0^{\mathcal{X}^k \mathcal{Y}}) = g_{\mathcal{Y}}(h(x_{kn}^{\mathcal{X}^k \mathcal{Y}}))$  which by (7.6) converges to  $\psi$  as  $k \rightarrow \infty$  (given that  $h(x_{kn}^{\mathcal{X}^k \mathcal{Y}})$  converges to the origin). Thus by (7.5),  $h(x_{kn}^{\mathcal{X}^k \mathcal{Y}}) \rightarrow J\psi$  as  $k \rightarrow \infty$ . But  $\psi \neq J\psi$ , for otherwise  $J$  would have an eigenvalue 1 which is not possible by (4.1). Thus the distance between  $h(x_0^{\mathcal{X}^k \mathcal{Y}})$  and  $h(x_n^{\mathcal{X}^k \mathcal{Y}})$  does not go to zero as  $k \rightarrow \infty$ . This a contradiction, hence  $\lambda_1 \lambda_2 = 1$  and  $\psi_2 \neq 0$ .

**Step 4.** Here we use the observation that points of  $\mathcal{X}^k \mathcal{Y}$ -cycles converge to those of the  $\mathcal{X}$ -cycle as  $k \rightarrow \infty$  to show that the  $\mathcal{X}$ -cycle is admissible and  $x_0^{\mathcal{X}} \notin \Sigma$ .

By (7.42) with  $j = \lfloor \frac{k}{2} \rfloor$ , the point  $h(x_{j_n}^{\mathcal{X}^k \mathcal{Y}})$  converges to the origin as  $k \rightarrow \infty$ . In  $x$ -coordinates this means that  $x_{j_n}^{\mathcal{X}^k \mathcal{Y}} \rightarrow x_0^{\mathcal{X}}$  as  $k \rightarrow \infty$ . Each  $x_{j_n}^{\mathcal{X}^k \mathcal{Y}}$  follows  $\mathcal{X}$  under  $f$  for sufficiently large values of  $k$  (by the admissibility of the  $\mathcal{X}^k \mathcal{Y}$ -cycles). Thus  $x_0^{\mathcal{X}}$  must also follow  $\mathcal{X}$  under  $f$  because  $f$  is continuous. That is, the  $\mathcal{X}$ -cycle is admissible.

By (7.38) and (7.42) with  $\lambda_1 \lambda_2 = 1$ ,

$$h(x_{(k-1)n}^{\mathcal{X}^k \mathcal{Y}}) \rightarrow \frac{\gamma_{12} \psi_2}{(1+c)\lambda_1} e_1, \quad h(x_{kn}^{\mathcal{X}^k \mathcal{Y}}) \rightarrow \frac{\gamma_{12} \psi_2}{1+c} e_1,$$

as  $k \rightarrow \infty$ . But  $h(x_{(k-1)n}^{\mathcal{X}^k \mathcal{Y}})$  and  $h(x_{kn}^{\mathcal{X}^k \mathcal{Y}})$  lie on different sides of  $h(\Sigma)$  for arbitrarily large values of  $k$ , so if  $h(\Sigma)$  passes through the origin (so that  $x_0^{\mathcal{X}} \in \Sigma$ ) it must contain the first coordinate axis. As in Step 3, this is not possible because  $e_1^\top \zeta_1 \neq 0$ . Therefore  $x_0^{\mathcal{X}} \notin \Sigma$ .

**Step 5.** Here we construct the homoclinic  $\mathcal{S}$ -orbit  $\{y_i\}$ . Our main efforts are in showing that  $y_\alpha \in \Sigma$ . For this step we work in  $x$ -coordinates because our arguments utilise the continuity of  $f$  and the formula (2.2).

Since  $\gamma_{11} = 0$  and  $\psi_1 = 0$ , by (7.6) we have  $e_1^\top g_{\mathcal{Y}}(u) = 0$  for any  $u = ae_1$ , where  $a \in \mathbb{R}$ . In  $x$ -coordinates this means that every  $x \in E^u(x_0^{\mathcal{X}})$  maps to  $E^s(x_0^{\mathcal{X}})$  under  $f_{\mathcal{Y}}$ .

Let  $y_0 = E^u(x_0^{\mathcal{X}}) \cap \Sigma$ , which is a unique point because  $e_1^\top \zeta_1 \neq 0$  by assumption. Since  $y_0 \in E^u(x_0^{\mathcal{X}})$ , we have  $f_{\mathcal{Y}}(y_0) \in E^s(x_0^{\mathcal{X}})$ . Therefore there exists an  $\mathcal{S}$ -orbit  $\{y_i\}$  involving  $y_0$  that



is homoclinic to the  $\mathcal{X}$ -cycle (although it may not be admissible). Also  $y_0 \in \Sigma$ , thus by Lemma 2.1,  $f_{\mathcal{Y}\bar{\sigma}}(y_0) = f_{\mathcal{Y}}(y_0)$  and so  $f_{\mathcal{Y}\bar{\sigma}}(y_0) \in E^s(x_0^{\mathcal{X}})$ . Then by assumption (4),  $y_0$  must be the only element of  $E^u(x_0^{\mathcal{X}})$  that maps to  $E^s(x_0^{\mathcal{X}})$  under  $f_{\mathcal{Y}\bar{\sigma}}$ .

Let  $\tilde{\mathcal{X}}$  denote the first  $\alpha$  elements of  $\mathcal{X}\mathcal{Y}$  and  $\tilde{\mathcal{Y}}$  denote the remaining elements of  $\mathcal{X}\mathcal{Y}$ . That is

$$\tilde{\mathcal{X}}\tilde{\mathcal{Y}} = \mathcal{X}\mathcal{Y}, \quad (7.43)$$

where  $\tilde{\mathcal{X}}$  is of length  $\alpha$  and  $\tilde{\mathcal{Y}}$  is of length  $n + p - \alpha$ . Since  $(\mathcal{X}\mathcal{Y})^{\bar{\sigma}\bar{\alpha}} = \mathcal{Y}\mathcal{X}$ ,

$$\tilde{\mathcal{X}}^{\bar{\sigma}}\tilde{\mathcal{Y}}^{\bar{\sigma}} = \mathcal{Y}\mathcal{X}. \quad (7.44)$$

Since  $y_0 \in \Sigma$ ,

$$y_\alpha = f_{\tilde{\mathcal{X}}}(y_0) = f_{\tilde{\mathcal{X}}\bar{\sigma}}(y_0), \quad (7.45)$$

by Lemma 2.1.

For any  $a \in \mathbb{R}$ , we have  $x_0^{\mathcal{X}} + a\zeta_1 \in E^u(x_0^{\mathcal{X}})$ . Consider the affine function

$$\phi_1(a) = e_1^\top f_{\tilde{\mathcal{X}}}(x_0^{\mathcal{X}} + a\zeta_1). \quad (7.46)$$

Below we show that  $\frac{d\phi_1}{da} \neq 0$ . Given that this is true, there exists unique  $\hat{a} \in \mathbb{R}$  such that  $\phi_1(\hat{a}) = 0$ . Let  $\hat{x} = x_0^{\mathcal{X}} + \hat{a}\zeta_1$ . Then  $f_{\tilde{\mathcal{X}}}(\hat{x}) \in \Sigma$ , and so

$$f_{\tilde{\mathcal{X}}\tilde{\mathcal{Y}}}(\hat{x}) = f_{(\tilde{\mathcal{X}}\tilde{\mathcal{Y}})^{\bar{\sigma}\bar{\alpha}}}(\hat{x}), \quad (7.47)$$

by Lemma 2.1. By  $\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\bar{\sigma}\bar{\alpha}}$  and (7.43), we can alter the symbols in (7.47) to produce

$$f_{\mathcal{X}\mathcal{Y}}(\hat{x}) = f_{\mathcal{Y}\bar{\sigma}\mathcal{X}}(\hat{x}). \quad (7.48)$$

Since  $\hat{x} \in E^u(x_0^{\mathcal{X}})$ , we have  $f_{\mathcal{X}}(\hat{x}) \in E^u(x_0^{\mathcal{X}})$  (because  $E^u(x_0^{\mathcal{X}})$  is invariant under  $f_{\mathcal{X}}$ ), and thus  $f_{\mathcal{X}\mathcal{Y}}(\hat{x}) \in E^s(x_0^{\mathcal{X}})$ . That is,  $f_{\mathcal{Y}\bar{\sigma}\mathcal{X}}(\hat{x}) \in E^s(x_0^{\mathcal{X}})$ , by (7.48). Thus  $f_{\mathcal{Y}\bar{\sigma}}(\hat{x}) \in E^s(x_0^{\mathcal{X}})$  (because  $E^s(x_0^{\mathcal{X}})$  is invariant under  $f_{\mathcal{X}}$ ). Above we showed that  $y_0$  is the only element of  $E^u(x_0^{\mathcal{X}})$  that maps to  $E^s(x_0^{\mathcal{X}})$  under  $f_{\mathcal{Y}\bar{\sigma}}$ , hence  $\hat{x} = y_0$ . Thus  $f_{\tilde{\mathcal{X}}}(\hat{x})$  is the point  $y_\alpha$ , and so  $y_\alpha \in \Sigma$ .

To complete this step we show that  $\frac{d\phi_1}{da} \neq 0$ . Consider the affine function

$$\phi_2(a) = \omega_1^\top (f_{\mathcal{Y}\bar{\sigma}\mathcal{X}}(x_0^{\mathcal{X}} + a\zeta_1) - x_0^{\mathcal{X}}). \quad (7.49)$$

This function returns a measure of the distance of the point  $f_{\mathcal{Y}\bar{\sigma}\mathcal{X}}(x_0^{\mathcal{X}} + a\zeta_1)$  from  $E^s(x_0^{\mathcal{X}})$ , where  $x_0^{\mathcal{X}} + a\zeta_1 \in E^u(x_0^{\mathcal{X}})$ . We have  $\phi_2(a_0) = 0$  because  $h(y_0) = a_0 e_1$  and so  $y_0 = x_0^{\mathcal{X}} + a_0 \zeta_1$ . Since  $y_0$  is the only element of  $E^u(x_0^{\mathcal{X}})$  that maps to  $E^s(x_0^{\mathcal{X}})$  under  $f_{\mathcal{Y}\bar{\sigma}}$ , and  $E^s(x_0^{\mathcal{X}})$  is invariant under  $f_{\mathcal{X}}$ , the value  $a_0$  is the unique zero of  $\phi_2(a)$ . Thus we must have  $\frac{d\phi_2}{da} \neq 0$ .

From (2.2) and (2.6) we obtain the formula

$$f_{\mathcal{Y}\bar{\sigma}\mathcal{X}}(x) = f_{(\mathcal{Y}\mathcal{X})^{\bar{\sigma}\bar{\alpha}}}(x) + \xi_{\tilde{\mathcal{Y}}} e_1^\top f_{\tilde{\mathcal{X}}}(x), \quad (7.50)$$

where

$$\xi_{\tilde{\mathcal{Y}}} = A_{\tilde{\mathcal{Y}}_{n+p-\alpha-1}} \cdots A_{\tilde{\mathcal{Y}}_1} \xi. \quad (7.51)$$

By using  $\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\bar{0}\bar{\alpha}}$ , we can therefore write

$$\phi_2(a) = \omega_1^\top (f_{\mathcal{X}\mathcal{Y}}(x_0^\mathcal{X} + a\zeta_1) - x_0^\mathcal{X} + \xi_{\bar{\mathcal{Y}}}\phi_1(a)) .$$

But  $\omega_1^\top (f_{\mathcal{X}\mathcal{Y}}(x_0^\mathcal{X} + a\zeta_1) - x_0^\mathcal{X})$  is identically zero because every  $x \in E^u(x_0^\mathcal{X})$  maps to  $E^s(x_0^\mathcal{X})$  under  $f_{\mathcal{Y}}$ , thus  $\phi_2(a) = \omega_1^\top \xi_{\bar{\mathcal{Y}}}\phi_1(a)$ . Therefore  $\frac{d\phi_1}{da} \neq 0$  because  $\frac{d\phi_2}{da} \neq 0$ .

**Step 6.** Next we use the assumption that  $\mathcal{X}^k\mathcal{Y}$ -cycles are admissible to show that  $c \geq \lambda_2$ , and use the assumption that  $\mathcal{X}^k\mathcal{Y}$ -cycles are stable to show that  $c \leq 1$ .

Equation (7.16) is a formula for  $\Gamma J^k$  obtained in the proof of Theorem 5.1. This formula is valid here because we have shown that  $\gamma_{11} = 0$ ,  $\psi_1 = 0$  and  $\lambda_1\lambda_2 = 1$ . As noted above, it can be seen from (7.16) that  $\Gamma J^k$  has an eigenvalue that converges to  $i\sqrt{c}$  as  $k \rightarrow \infty$ . The stability of the  $\mathcal{X}^k\mathcal{Y}$ -cycles requires all eigenvalues of  $\Gamma J^k$  to have modulus less than or equal to 1, hence we must have  $|c| \leq 1$ .

Equation (7.42) with  $j = k$  gives

$$h\left(x_{kn}^{\mathcal{X}^k\mathcal{Y}}\right) = \frac{\gamma_{12}\psi_2}{1+c} e_1 + \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) + \mathcal{O}(\lambda_1^k\beta^k) ,$$

where we have substituted (7.38) and  $\lambda_1\lambda_2 = 1$  (and recall  $c \neq -1$ , by assumption). By then substituting (7.8) (since we have shown that  $a_0e_1 = h(y_0)$  follows both  $\mathcal{X}\mathcal{Y}$  and  $\mathcal{Y}\mathcal{X}$  under  $g$ ) with  $\lambda_1\lambda_2 = 1$  and (7.37), we obtain

$$h\left(x_{kn}^{\mathcal{X}^k\mathcal{Y}}\right) = \frac{a_0(\lambda_1 + 1)c}{1+c} e_1 + \mathcal{O}\left(\frac{1}{\lambda_1^k}\right) + \mathcal{O}(\lambda_1^k\beta^k) .$$

By the admissibility of the  $\mathcal{X}^k\mathcal{Y}$ -cycles, and the admissibility of the  $\mathcal{X}$ -cycle (see Step 4), we know that the points  $x_0^\mathcal{X}$  and  $x_{kn}^{\mathcal{X}^k\mathcal{Y}}$  lie on different sides of  $\Sigma$ . Since  $h(x_0^\mathcal{X})$  is the origin and  $a_0e_1 \in h(\Sigma)$ , we must have  $\frac{(\lambda_1+1)c}{1+c} \geq 1$ . This implies  $c \geq \lambda_2$  (using  $\lambda_1\lambda_2 = 1$  and  $c > -1$ ).

**Step 7.** Finally we establish the properties of the  $\mathcal{X}^k\mathcal{Y}^{\bar{0}}$ -cycles.

We write the map  $g_{\mathcal{Y}^{\bar{0}}}$  as

$$g_{\mathcal{Y}^{\bar{0}}}(u) = \Xi u + \chi , \tag{7.52}$$

where  $\Xi = Q^{-1}M_{\mathcal{Y}^{\bar{0}}}Q$  and  $\chi = Q^{-1}(P_{\mathcal{Y}^{\bar{0}}}b - (I - M_{\mathcal{Y}^{\bar{0}}})x_0^\mathcal{X})$ . For each  $i, j = 1, \dots, N$ , we let  $\xi_{ij}$  denote the  $(i, j)$ -element of  $\Xi$  and  $\chi_i = e_i^\top \chi$ . Assumption (4), which states that  $f_{\mathcal{Y}^{\bar{0}}}$  does not map  $E^u(x_0^\mathcal{X})$  to  $E^s(x_0^\mathcal{X})$ , implies that  $e_1^\top g_{\mathcal{Y}^{\bar{0}}}(ae_1)$  is nonzero for some  $a \in \mathbb{R}$ . But  $e_1^\top g_{\mathcal{Y}^{\bar{0}}}(a_0e_1) = 0$  because  $y_0 = h^{-1}(a_0e_1)$  maps to  $E^s(x_0^\mathcal{X})$  under  $f_{\mathcal{Y}}$  and  $y_0 \in \Sigma$ . By (7.52), we have  $e_1^\top g_{\mathcal{Y}^{\bar{0}}}(ae_1) = \xi_{11}a + \chi_1$ . Thus we must have  $\xi_{11} \neq 0$  and

$$\chi_1 = -a_0\xi_{11} . \tag{7.53}$$

By (7.2) we obtain

$$\det(I - \Xi J^k) = -\xi_{11}\lambda_1^k + \mathcal{O}(1) , \tag{7.54}$$

which is nonzero for sufficiently large values of  $k$ . Therefore  $\mathcal{X}^k\mathcal{Y}^{\bar{0}}$ -cycles exist and are unique (although may not be admissible) for sufficiently large values of  $k$ , and we denote them  $\{x_i^{\mathcal{X}^k\mathcal{Y}^{\bar{0}}}\}$ .

The point  $h\left(x_{kn}^{\mathcal{X}^k\mathcal{Y}^0}\right)$  is a fixed point of  $g_{\mathcal{Y}^0\mathcal{X}^k}(u) = J^k(\Xi u + \chi)$ . By using (7.2) and (7.53) to solve this fixed point equation, we obtain

$$h\left(x_{kn}^{\mathcal{X}^k\mathcal{Y}^0}\right) = a_0 e_1 + \mathcal{O}\left(\frac{1}{\lambda_1^k}\right), \quad (7.55)$$

and hence  $x_{kn}^{\mathcal{X}^k\mathcal{Y}^0} \rightarrow y_0$  as  $k \rightarrow \infty$ . ■

## 8 Discussion

This paper provides rigorous results for piecewise-linear continuous maps (2.1) of any number of dimensions. From a bifurcation theory perspective, results of such generality are relatively rare because one cannot usually reduce the dimensionality of (2.1) through a centre manifold analysis. Other rigorous results for (2.1) with an arbitrary number of dimensions include the border-collision normal form [29], the characterisation of fixed points and period-two solutions [30, 31], invariant measures and  $N$ -dimensional attractors [32, 33], and shrinking points of mode-locking regions [27, 28].

Theorems 5.1 and 5.2 provide an equivalence between subsumed homoclinic connections and infinitely many asymptotically stable periodic solutions in a codimension-three scenario. We have chosen to state the two results separately as they provide considerably more information than their combination as a single “if and only if” theorem. With multiple attractors the long-term dynamics of  $f$  depends on the initial conditions and can be a form of unpredictability if the boundaries of the basins of attraction are complicated. The presence of many attractors indicates a high level of complexity and is important in diverse areas of application [34].

Given a multi-parameter collection of maps of the form (2.1), a practical approach for finding such codimension-three scenarios is to numerically solve the three codimension-one conditions (6.2). Once this is achieved, one can check that all the conditions in Theorems 5.1 and 5.2 are satisfied. Section 6 provided three examples with parameter values accurate to ten decimal places. In each case we showed the  $\mathcal{X}^k\mathcal{Y}$ -cycles for the smallest eight values of  $k$  for which they are admissible and asymptotically stable.

For each of the examples in §6 (and those in [22]), we have  $d \neq 0$  and  $\gcd(d, n) = 1$  (3.5). Yet Theorems 5.1 and 5.2 involve no such restriction on the value of  $d$ . It remains to be determined if the codimension-three scenario is feasible when these constraints on  $d$  are not satisfied. In such cases the subsumed homoclinic connection would not connect the points of the  $\mathcal{X}$ -cycle with a single closed loop. It also remains to identify the codimension-three scenario in a system of differential equations. For instance, it has recently been shown that multiple attractors can be created in grazing-sliding bifurcations of piecewise-smooth systems of ODEs (which can be viewed as special types of border-collision bifurcations). However, currently it is not known how many attractors be created [35, 36].

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